Hadronic light-by-light contribution to the muon anomalous magnetic moment from lattice QCD

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*Moriond Electroweak 2019*

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1. Introduction and background

2. Hadronic light-by-light (HLbL) scattering contribution

3. Results for QED$_L$

4. Results for QED$_\infty$

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\[ \langle \mu(p')|J_{\nu}(0)|\mu(p)\rangle = -e\bar{u}(p') \left( F_1(q^2)\gamma_{\nu} + i \frac{F_2(q^2)}{4m} [\gamma_{\nu}, \gamma_\rho] q_\rho \right) u(p) \]

\[ a_\mu \equiv (g - 2)/2 = F_2(0) \quad (q = p' - p) \]
### Experiment - Theory

<table>
<thead>
<tr>
<th>SM Contribution</th>
<th>Value ± Error ($\times 10^{11}$)</th>
<th>Ref</th>
<th>notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED (5 loops)</td>
<td>116584718.951 ± 0.080</td>
<td>[Aoyama et al., 2012]</td>
<td></td>
</tr>
<tr>
<td>HVP LO</td>
<td>6931 ± 34</td>
<td>[Davier et al., 2017]</td>
<td>$\rightarrow 3.5\sigma$</td>
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<tr>
<td></td>
<td>6932.6 ± 24.6</td>
<td>[Keshavarzi et al., 2018]</td>
<td>$\rightarrow 3.7\sigma$</td>
</tr>
<tr>
<td></td>
<td>6925 ± 27</td>
<td>[Blum et al., 2018]</td>
<td>lattice+R-ratio (FJ17), $\rightarrow 3.7\sigma$</td>
</tr>
<tr>
<td>HVP NLO</td>
<td>$-98.2 \pm 0.4$</td>
<td>[Keshavarzi et al., 2018]</td>
<td></td>
</tr>
<tr>
<td>HVP NNLO</td>
<td>12.4 ± 0.1</td>
<td>[Kurz et al., 2014]</td>
<td></td>
</tr>
<tr>
<td>HLbL</td>
<td>105 ± 26</td>
<td>[Prades et al., 2009]</td>
<td></td>
</tr>
<tr>
<td>HLbL (NLO)</td>
<td>3 ± 2</td>
<td>[Colangelo et al., 2014]</td>
<td></td>
</tr>
<tr>
<td>Weak (2 loops)</td>
<td>153.6 ± 1.0</td>
<td>[Gnendiger et al., 2013]</td>
<td></td>
</tr>
<tr>
<td>SM Tot</td>
<td>116591820.5 ± 35.6</td>
<td>[Keshavarzi et al., 2018]</td>
<td></td>
</tr>
<tr>
<td>Exp (0.54 ppm)</td>
<td>116592080 ± 63</td>
<td>[Bennett et al., 2006]</td>
<td></td>
</tr>
<tr>
<td>Diff (Exp − SM)</td>
<td>259.5 ± 72</td>
<td>[Keshavarzi et al., 2018]</td>
<td>$\rightarrow 3.7\sigma$</td>
</tr>
</tbody>
</table>

**main messages:** QCD errors dominate, $\Delta$ HLbL $\sim$ $\Delta$ HVP, discrepancy is large

FNAL E989 running, goal to reduce BNL 821 error by 1/4
Summary of HVP theory results (lattice and dispersive)  

- No new physics

**Lattice QCD**
- Fermilab/HPQCD /MILC 2019
- ETMC, 1808.00887
- RBC/UKQCD 1801.07224
- BMW, 1711.04980
- Mainz/CLS, 1705.01775
- HPQCD/RV 1601.03071

**Pheno.**
- $e^+e^-$
- $e^+e^-$
- $e^+e^- + \tau$
- $e^+e^- + \tau$

**Table VI.** Individual flavor contributions to the leading Taylor coefficients of the vacuum polarization function and the disconnected diagrams. Contributions are from the lattice analysis; the second comes from uncertainties on finer lattices. Analysis on QCD+QED gluon field ensembles will be important here too to take into account finite-volume uncertainty. Further details of the QED procedure are provided as supplementary material.

**Analysis and Results**

- Work on all of these is in progress.

**Figure 4.** Comparison of the R-ratio $R_{\text{Light+Strange}}$ for the 64I ensemble. The solid line is the HVP lattice result as a finite-volume error. The dashed line is the improved dispersive result as a finite-volume error.

Lattice and dispersive results agree well [Blum et al., 2018]
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The desired amplitude

\[
\langle \mu(x_{\text{snk}}) J_\nu(x_{\text{op}}) \bar{\mu}(x_{\text{src}}) \rangle = -eM_\nu(x_{\text{src}}, x_{\text{op}}, x_{\text{snk}})
\]

is obtained from a Euclidean space correlation function computed on the lattice

\[
\left[ \left( \frac{-i \not{q}^+ + m_\mu}{2E_{q/2}} \right) F_1(q^2) \gamma_\nu + i \frac{F_2(q^2)}{4m} [\gamma_\nu, \gamma_\rho] q_\rho \right] \left( \frac{-i \not{q}^- + m_\mu}{2E_{q/2}} \right) \right]_{\alpha\beta} = \left( M_\nu(\vec{q}) \right)_{\alpha\beta},
\]
Point source method in QCD+pQED (L. Jin) [Blum et al., 2016]

- Point source propagators at $x$ and $y$
- Important sampling is used to choose source locations:
  - Most of the contribution from $|x - y| \lesssim 1$ fm
  - Compute for all possible $|x - y| < r = 4 - 5$ lattice units
  - Randomly choose pairs for $|x - y| > r$
- Moment method allows computation of $F_2(q^2)$ directly at $q = 0$

Techniques produce $O(1000)$ improvement in statistical error over original method [Blum et al., 2015]
Lattice setup

- Photons: Feynman gauge, $QED_L$ [Hayakawa and Uno, 2008] (omit all modes with $\vec{q} = 0$)
- Gluons: Iwasaki (I) gauge action (RG improved, plaquette+rectangle)
- Muons: $L_s = \infty$ free domain-wall fermions (DWF)
- Quarks: Möbius-DWF

2+1f Möbius-DWF, I and I-DSDR physical point QCD ensembles (RBC/UKQCD) [Blum et al., 2014]

<table>
<thead>
<tr>
<th></th>
<th>48I</th>
<th>64I</th>
<th>24D</th>
<th>32D</th>
<th>32D fine</th>
<th>48D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a^{-1}$ (GeV)</td>
<td>1.73</td>
<td>2.36</td>
<td>1.0</td>
<td>1.0</td>
<td>1.38</td>
<td>1.0</td>
</tr>
<tr>
<td>$a$ (fm)</td>
<td>0.114</td>
<td>0.084</td>
<td>0.2</td>
<td>0.2</td>
<td>0.14</td>
<td>0.2</td>
</tr>
<tr>
<td>$L$ (fm)</td>
<td>5.47</td>
<td>5.38</td>
<td>4.8</td>
<td>6.4</td>
<td>4.6</td>
<td>9.6</td>
</tr>
<tr>
<td>$L_s$</td>
<td>48</td>
<td>64</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>$m_\pi$ (MeV)</td>
<td>139</td>
<td>135</td>
<td>140</td>
<td>140</td>
<td>140</td>
<td>140</td>
</tr>
<tr>
<td>$m_\mu$ (MeV)</td>
<td>106</td>
<td>106</td>
<td>106</td>
<td>106</td>
<td>106</td>
<td>106</td>
</tr>
<tr>
<td>meas (con,disco)</td>
<td>65,65</td>
<td>43,44</td>
<td>87,80</td>
<td>64,68</td>
<td>32,31</td>
<td>62,0</td>
</tr>
</tbody>
</table>
Continuum and $\infty$ volume limits in QED [Blum et al., 2016]

Test method in pure QED. QED systematics large, $O(a^4)$, $O(1/L^2)$, but under control

\begin{align*}
F_2(a, L) &= F_2 \left(1 - \frac{b_1}{(m_\mu L)^2} + \frac{b_2}{(m_\mu L)^4}\right) (1 - c_1 a^2 + c_2 a^4) \rightarrow F_2 = 46.6(2) \times 10^{-10}
\end{align*}

$O(1/L^2)$ finite volume (FV) error (c.f. exponentially suppressed in QCD)

Compare to analytic result, $46.5 \times 10^{-10}$
Disconnected contributions

![Diagram of disconnected contributions]

Leading \(O(m_s - m_{u,d})\)

\[O(m_s - m_{u,d})^2\] and higher

- Only 1 diagram does not vanish in SU(3) flavor limit
- Permutations of internal photons not shown
- Gluons within and connecting quark loops not shown
- To ensure loops are connected by gluons, explicit "vacuum" subtraction is required
We use two point sources at $y$ and $z$, chosen randomly. The point sinks $x_{op}$ and $x$ are summed over exactly on lattice.

- Only point source quark propagators are needed. We compute $M$ point source propagators and all $M^2$ combinations are used to perform the stochastic sum over $r = z - y$ ($M^2$ trick).

- Because of parity, expectation value for (moment of) left loop averages to zero.
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HLbL contribution, 139 MeV pion, \( a = 0.114 \text{ fm}, L = 5.5 \text{ fm} \) \cite{Blum2017a}

\[ a_{\mu}^{cHLbL} = (11.60 \pm 0.96) \times 10^{-10} \]

\[ a_{\mu}^{dHLbL} = (-6.25 \pm 0.80) \times 10^{-10} \]

\[ a_{\mu}^{HLbL} = (5.35 \pm 1.35) \times 10^{-10} \]

Need to extrapolate to the continuum and \( \infty \) volume limits
Lattices used for continuum and infinite volume extrapolation (from L. Jin)

48I: $48^3 \times 96$, 5.5fm box

64I: $64^3 \times 128$, 5.5fm box

24D: $24^3 \times 64$, 4.8fm box

32Dfine: $32^3 \times 64$, 4.8fm box

32D: $32^3 \times 64$, 6.4fm box
Continuum and infinite volume extrapolation  

\[ F_2(a, L) = F_2(1 - c_1 (m \mu L)^2)(1 - c_2 a^2) \]

\[ a^c_{\mu HLbL} = (28.46 \pm 3.66_{\text{stat}} \pm 0.28_{\text{sys,a^2}}) \times 10^{-10} \]

\[ a^d_{\mu HLbL} = -12.39 \pm 2.09_{\text{stat}} \pm 1.63_{\text{sys,a^2}} \times 10^{-10} \]

\[ a^H_{\mu LbL} = 16.06 \pm 3.90_{\text{stat}} \pm 1.91_{\text{sys,a^2}} \times 10^{-10} \]
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Infinite volume QED$_\infty$ [Green et al., 2015, Asmussen et al., 2016, Lehner and Izubuchi, 2015, Jin et al., 2015, Blum et al., 2017b]

Compute QCD part in finite volume, QED in $\infty$ volume

- Mainz group made first concrete proposal for QED$_\infty$
- QED$_\infty$: muon, photons computed in infinite volume, continuum (c.f. HVP)
- Leading FV error is exponentially suppressed (c.f. HVP) instead of $O(1/L^2)$
  - QCD mass gap: $\mathcal{H}(x, y, z, x_{op}) \sim \exp -m_\pi \times \text{dist}(x, y, z, x_{op})$
  - QED weight function does not grow exponentially
\( QED_\infty \) results- pure QED, lattice-spacing error [Blum et al., 2017b]

\[
\frac{F_2}{(\alpha/\pi)^3} = 0.3686(37)(35) \text{ and } 0.1232(30)(28) \text{ compared to QED perturbation theory results: } 0.371 \text{ and } 0.120
\]
cHLbL, QED$_\infty$, 139 MeV pion, $a = 0.2$ fm, $L = 6.4$ fm (preliminary)

Noisier than QED$_L$

![Graph showing Pion TFF, Lattice 32D, and Combine graphs with error bars]

Combine lattice (short distance) up to $R_{max}$ and pion (LMD) model (long distance) $R_{max}$ to $\infty$ for most precise result.
dHLbL, $\text{QED}_\infty$ (non-leading diagram), $m_\pi = 139$ MeV, $a = 0.2$ fm (preliminary)

result is negligible compared to error on leading contributions
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Hadronic light-by-light summary and outlook

- Lattice QCD(+QED) calculations done with physical masses, large boxes + improved measurement algorithms
- Physical point calculations published at $a = 0.114$ fm, 5.5 fm box [Blum et al., 2017a]
- Preliminary $a \to 0$, $L \to \infty$ limits taken in QED$_L$
  - connected, disconnected significant corrections
  - non-leading disconnected diagram makes small contribution
  - improving statistics
  - consistent with model, dispersive results.

- QED$_\infty$ noisier, $a \to 0$, $L \to \infty$ (QCD) limits in progress
- unlikely that HLbL contribution will rescue standard model

**Muon g-2 Theory Initiative** aims to have white paper this year, before E989 announces first results
Acknowledgments

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Complete Tenth-Order QED Contribution to the Muon g-2.

Position-space approach to hadronic light-by-light scattering in the muon $g - 2$ on the lattice.
*PoS, LATTICE2016:164.*

Bennett, G. et al. (2006).
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Calculation of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment.
*to be published, Phys. Rev. Lett.*
Hadronic light-by-light scattering contribution to the muon anomalous magnetic moment from lattice QCD.


Using infinite volume, continuum QED and lattice QCD for the hadronic light-by-light contribution to the muon anomalous magnetic moment.

Blum, T. et al. (2014).  
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Remarks on higher-order hadronic corrections to the muon $g-2$.  

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Towards the large volume limit - A method for lattice QCD + QED simulations.