Hamiltonian vs stability in alternative theories of gravity

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Rencontres de Moriond
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Introduction: \((A \Rightarrow B)\) does not mean \((\neg A \Rightarrow \neg B)\)

- **Known theorem**

  Hamiltonian density bounded by below
  \[\Rightarrow\] lowest energy state is stable

- **Converse statement often assumed**

  Hamiltonian density unbounded from below
  \[\Rightarrow\] unstable solution

Actually used several times in the literature:

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  \[ \Rightarrow \text{lowest energy state is stable} \]

- **Converse statement often assumed**

  **Hamiltonian density unbounded from below**
  \[ \Rightarrow \text{unstable solution} \]

Actually used several times in the literature:

Causal cones

In alternative theories of gravity, there exist generically different causal cones for perturbations around a given background.

Simplest example: k-essence

\[ \mathcal{L} = -\frac{1}{4} f(X), \quad \text{with} \quad X \equiv g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \]

Let
\[ \varphi = \bar{\varphi} + \chi \]
full field \quad background \quad perturbation

Second-order expansion of Lagrangian:

\[ \mathcal{L}_2 = -\frac{1}{2} G^{\mu\nu} \partial_\mu \chi \partial_\nu \chi, \quad \text{with} \quad G^{\mu\nu} = f'(\bar{X}) g^{\mu\nu} + 2 f''(\bar{X}) \nabla^\mu \bar{\varphi} \nabla^\nu \bar{\varphi} \]

effective metric
Causal cones (continued)

\[ \mathcal{L}_2 = -\frac{1}{2} \, g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \]

Effective metric

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**Hamiltonian**

\[ \mathcal{L}_2 = -\frac{1}{2} G^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \]

Effective metric

Conjugate momentum:

\[ p \equiv \frac{\partial \mathcal{L}_2}{\partial \dot{\chi}} = -G^{00} \dot{\chi} - G^{0i} \partial_i \chi \]

Hamiltonian density:

\[ \mathcal{H}_2 = p \dot{\chi} - \mathcal{L}_2 = -\frac{1}{2} G^{00} (p + G^{0i} \partial_i \chi)^2 + \frac{1}{2} G^{ij} \partial_i \chi \partial_j \chi \]

**Hamiltonian bounded by below**

\[ \mathcal{H}_2 \geq 0 \iff G^{00} < 0 \text{ and } G^{ij} \text{ positive} \]

**Hyperbolicity in the \((t, x)\) subspace**

\[ G^{\mu\nu} = \begin{pmatrix} G^{00} & G^{0x} \\ G^{0x} & G^{xx} \end{pmatrix} \] with determinant \( D = G^{00} G^{xx} - (G^{0x})^2 < 0 \)
Hamiltonian

\[ \mathcal{L}_2 = -\frac{1}{2} G^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \]

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Compare to

Hyperbolicity in the \((t, x)\) subspace

\[ G^{\mu\nu} = \begin{pmatrix} G^{00} & G^{0x} \\ G^{0x} & G^{xx} \end{pmatrix} \quad \text{with determinant} \quad D \equiv G^{00} G^{xx} - (G^{0x})^2 < 0 \]
Hamiltonian for a tilted scalar causal cone

Hyperbolicity in the \((t, x)\) subspace

\[
\mathcal{G}_{\mu\nu} = \begin{pmatrix}
\mathcal{G}^{00} & \mathcal{G}^{0x} \\
\mathcal{G}^{0x} & \mathcal{G}^{xx}
\end{pmatrix}
\]

with determinant \(D \equiv \mathcal{G}^{00}\mathcal{G}^{xx} - (\mathcal{G}^{0x})^2 < 0\)

Let \(\mathcal{G}_{\mu\nu}\) denote the inverse effective metric \([\neq \mathcal{G}_{\mu\lambda}\mathcal{G}_{\nu\rho}\mathcal{G}^{\lambda\rho}]\)

In the \((t, x)\) subspace:

\[
\begin{pmatrix}
\mathcal{G}^{00} & \mathcal{G}^{0x} \\
\mathcal{G}^{0x} & \mathcal{G}^{xx}
\end{pmatrix} = \begin{pmatrix}
\mathcal{G}^{xx} & -\mathcal{G}^{0x} \\
-\mathcal{G}^{0x} & \mathcal{G}^{00}
\end{pmatrix} / D
\]

If time axis outside scalar cone:

\(\mathcal{G}^{00} dt dt > 0 \Rightarrow \mathcal{G}^{xx} < 0\)

\(\Rightarrow\) Hamiltonian density unbounded by below!

\[
\mathcal{H}_2 = -\frac{1}{2\mathcal{G}^{00}} (p + \mathcal{G}^{0i}\partial_i\chi)^2 + \frac{1}{2} \mathcal{G}^{ij}\partial_i\chi\partial_j\chi
\]
Hamiltonian is coordinate dependent

\[ \mathcal{H}_2 \geq 0 \]

\[ \mathcal{H}_2 \text{ unbounded by below} \]

This is strictly the *same* theory, viewed by a moving observer.

⇒ Must be stable, in spite of Hamiltonian unboundedness!
Superluminal case

If $x$ axis inside scalar cone: $\mathcal{G}_{xx} \, dx \, dx < 0 \Rightarrow \mathcal{G}^{00} > 0$

because $\begin{pmatrix} \mathcal{G}_{00} & \mathcal{G}_{0x} \\ \mathcal{G}_{0x} & \mathcal{G}_{xx} \end{pmatrix} = \begin{pmatrix} \mathcal{G}^{xx} & -\mathcal{G}^{0x} \\ -\mathcal{G}^{0x} & \mathcal{G}^{00} \end{pmatrix} / D$

$\Rightarrow$ Hamiltonian density also unbounded by below

$$\mathcal{H}_2 = -\frac{1}{2\mathcal{G}^{00}} \left( p + \mathcal{G}^{0i} \partial_i \chi \right)^2 + \frac{1}{2} \mathcal{G}^{ij} \partial_i \chi \partial_j \chi$$
Relative orientations of two causal cones
Conserved quantities

- $\mathcal{L}_2 = -\frac{1}{2} G^{\mu\nu} \partial_\mu \chi \partial_\nu \chi$ is diffeomorphism-invariant because $G^{\mu\nu}$ is a tensor

- $-T^\nu_\mu \equiv \frac{\delta \mathcal{L}_2}{\delta (\partial_\nu \chi)} \partial_\mu \chi - \delta^\nu_\mu \mathcal{L}_2$ is thus conserved:

$$\partial_\nu T^\nu_\mu = 0 \iff \partial_0 T^0_\mu + \partial_i T^i_\mu = 0 \quad \text{[flat-spacetime case to simplify]}$$

Let $P_\mu \equiv - \iiint_V T^0_\mu \, d^3x$ \implies $\partial_t P_\mu = 0$

In particular, energy conservation $\partial_t P_0 = 0$

where $-T^0_0$ = on-shell value of the Hamiltonian density

- Coordinate change $P'_\lambda = (\partial x^\mu / \partial x'^\lambda) P_\mu$

- Even if $P_0 \geq 0$, this is not always so for $P'_0 = (\partial x^\mu / \partial x'^0) P_\mu$

$$[= (P_0 + \nu P_x) / \sqrt{1 - \nu^2} \text{ for a mere boost}]$$

- But all $P'_\mu$ are conserved, and the linear combination $P_0 = (\partial x^\mu / \partial x'^0) P'_\mu$ is bounded by below \implies stable!
Conserved quantities

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\]
Relative orientations of two causal cones
Conditions on effective metric $\mathcal{G}^{\mu\nu}$ for stability

Choose coordinates such that $g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in the $(t, x)$ subspace

**Stability needs**

- **Hyperbolicity:**
  \[ D \equiv \mathcal{G}^{00}\mathcal{G}^{xx} - (\mathcal{G}^{0x})^2 < 0 \]

- $\exists$ consistent time and space coordinates:
  \[ \mathcal{G}^{00} < \mathcal{G}^{xx} \text{ and/or } |\mathcal{G}^{00} + \mathcal{G}^{xx}| < 2|\mathcal{G}^{0x}| \]

$\Rightarrow$ off-diagonal component $\mathcal{G}^{0x}$ should be large enough

whereas **sign** of Hamiltonian density does not depend on it:

\[ \mathcal{H}_2 = -\frac{1}{2\mathcal{G}^{00}} (p + \mathcal{G}^{0i} \partial_i \chi)^2 + \frac{1}{2} \mathcal{G}^{ij} \partial_i \chi \partial_j \chi \]
Covariant conditions for stability

**Weak Energy Condition in general relativity**

∀ timelike $U^\mu$, $T_{\mu\nu}U^\mu U^\nu \geq 0$

But what does “timelike” mean when there are different causal cones?

Here, we need:

- *All* effective metrics ($g^{\mu\nu}$, $G^{\mu\nu}$, ...) are of $(-, +, +, +)$ signature
- ∃ contravariant vector $U^\mu$ and covariant vector $u_\mu$ such that $g_{\mu\nu}U^\mu U^\nu < 0$, $G_{\mu\nu}U^\mu U^\nu < 0$
  (∃ common interior to all causal cones: $dx^0$ in the direction of $U^\mu$)
- $g^{\mu\nu}u_\mu u_\nu < 0$, $G^{\mu\nu}u_\mu u_\nu < 0$
  (∃ spatial hypersurface exterior to all cones: $u_\mu dx^\mu = 0$)
- and $T^\nu_{\mu}U^\mu u_\nu \geq 0$
  (positivity of Hamiltonian in a good coordinate system)
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  g_{\mu\nu} U^\mu U^\nu < 0, \quad G_{\mu\nu} U^\mu U^\nu < 0
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- $g^{\mu\nu} u_\mu u_\nu < 0, \quad G^{\mu\nu} u_\mu u_\nu < 0$
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- and
  
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  T^\nu_{\mu} U^\mu u_\nu \geq 0
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  (positivity of Hamiltonian in a good coordinate system)
Example of k-essence

Simplest example: k-essence

\[ \mathcal{L} = -\frac{1}{4} f(X), \quad \text{with} \quad X \equiv g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \]

Well-posed Cauchy problem (hyperbolic equations) and stability

\[ \Leftrightarrow \quad f'(X) > 0 \quad \text{and} \quad 2X f''(X) + f'(X) > 0 \]

(But no condition on \( f''(X) \) alone:

\( f''(X) < 0 \) infraluminal scalar,

\( f''(X) > 0 \) superluminal scalar)

A simple self-tuning model

\[ S = \int \sqrt{-g} \, d^4x \left[ \zeta (R - 2\Lambda_{\text{bare}}) - \eta (\partial_\mu \varphi)^2 + \beta G^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right] \]

Exact Schwarzschild-de Sitter solution:

\[ ds^2 = -A(r) \, dt^2 + \frac{dr^2}{A(r)} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \]

\[ A(r) = 1 - \frac{2Gm}{r} - \frac{\Lambda_{\text{eff}}}{3} r^2, \quad \text{with} \quad \Lambda_{\text{eff}} = -\frac{\eta}{\beta} \]

\[ \varphi = q \left[ t - \int \frac{\sqrt{1 - A(r)}}{A(r)} \, dr \right], \quad \text{with} \quad q^2 = \frac{\eta + \beta \Lambda_{\text{bare}}}{\eta \beta} \zeta \]

[E. Babichev & C. Charmousis, JHEP 1408 (2014) 106]

(generalized in [E. Babichev & GEF, Phys. Rev. D 95 (2017) 024020])
A simple self-tuning model

“John” ∈ “Fab Four” model ⊂ Horndeski class

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Graviton causal cone

Diagonalize the spin-2 and spin-0 kinetic terms around an arbitrary background ("Einstein frame")?

**Well-known for \( f(R) \) theories**

\[
S = \int d^4x \sqrt{-g} f(R) \Leftrightarrow \int d^4x \sqrt{-g} \left\{ f(\phi) + (R - \phi) f'(\phi) \right\}
\]

Let \( g_{\mu\nu}^* \equiv f'(\phi) g_{\mu\nu} \) and \( \varphi \equiv \frac{\sqrt{3}}{2} \ln f'(\phi) \) ⇒ standard scalar-tensor theory

**Quadratic + cubic Galileon**

\[
S = \frac{1}{4\pi G} \int d^4x \sqrt{-g} \left\{ \frac{R}{4} - \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{k_3}{2} (\partial_\mu \varphi)^2 \square \varphi \right\}
\]

Redefine \( h_{\mu\nu}^* \equiv h_{\mu\nu} + 4k_3 \left[ \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} (\partial_\lambda \varphi)^2 \right] \chi \)


For the "John" model, we did not find any covariant diagonalization!
⇒ Study odd-parity perturbations ⇒ effective metric for gravitons

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For the “John” model, we did not find any covariant diagonalization!

$\Rightarrow$ Study odd-parity perturbations $\Rightarrow$ effective metric for gravitons

Matter and graviton causal cones

Close to black-hole horizon

Close to cosmological horizon
Scalar causal cone

But this does not suffice to prove stability:
The scalar causal cone should also be consistent with the graviton causal cone

⇒ Study \( \ell = 0 \) even-parity modes, to extract the effective metric for the scalar degree of freedom

In the asymptotic de Sitter Universe, the solution is stable if

\[
\text{either } \eta > 0, \quad \beta < 0 \quad \text{and} \quad \frac{\Lambda_{\text{bare}}}{3} < -\frac{\eta}{\beta} < \Lambda_{\text{bare}} \\
\text{or } \eta < 0, \quad \beta > 0 \quad \text{and} \quad \Lambda_{\text{bare}} < -\frac{\eta}{\beta} < 3\Lambda_{\text{bare}}
\]

What happens in the vicinity of the black hole?
Scalar causal cone

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What happens in the vicinity of the black hole?
Graviton, scalar and matter causal cones
Conclusions

- A Hamiltonian density which is unbounded from below does not always imply an instability!

- 3-momentum is also conserved, and can be combined with Hamiltonian to give a bounded-by-below quantity (= Hamiltonian in another coordinate system).

- Simplest way of analyzing a generic case: Plot the causal cones for each degree of freedom. There should exist both a common interior and a common exterior spacelike hypersurface.

- ∃ generalization of the Weak Energy Condition of general relativity to encompass the case of several causal cones.

- An exact Schwarzschild-de Sitter solution of a Horndeski theory is stable for a given range of its parameters, contrary to a claim in the literature.
Graviton speed

In the asymptotic de Sitter solution, \( c_{\text{gravity}} \neq c_{\text{light}} \)

Inconsistent with GW170817 event


Couple matter to the disformal metric

\[
\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{\beta}{\zeta + \frac{\beta}{2}(\partial_\lambda \varphi)^2} \partial_\mu \varphi \partial_\nu \varphi = \left(1 + \frac{\Lambda_{\text{bare}}}{\Lambda_{\text{eff}}}\right) g_{\mu\nu}^{\text{graviton}}
\]

i.e., write \( S_{\text{matter}}[\text{matter fields; } \tilde{g}_{\mu\nu}] \)
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i.e., write \( S_{\text{matter}} \) \([\text{matter fields}; \tilde{g}_{\mu\nu}]\)
\[ c_{\text{gravity}} = c_{\text{light}} \text{ everywhere!} \]
Graviton and scalar causal cones

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