A New Family Symmetry: Discrete Quaternion Group

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M.F., Satoru Kaneko, Ernest Ma and Morimitsu Tanimoto,
QUATERNION FAMILY SYMMETRY OF QUARKS AND LEPTONS
A motivation: the form of $M_\nu$

- What we know on neutrinos: $\Delta m_{\text{sol}}^2$ and $\theta_{12}$, $\Delta m_{\text{atm}}^2$ and $\theta_{23}$, upper bound on $m_i$ and $\theta_{13}$.
- All these data (as well as CP phases) are encoded in the structure of the 3x3 Majorana neutrino mass matrix $M_\nu$.
- Many viable ideas on the possible structure of $M_\nu$: quasi-degeneracy, $\mu \leftrightarrow \tau$ symmetry, bimaximal mixing, dominant $\mu \tau$-block, flavor democracy, texture zeros …
- Search for the underlying family symmetry, if any.

Why to consider a quaternion symmetry?
What is the corresponding form of $M_\nu$?
Data on neutrino oscillations

Assuming normal ordering of the mass spectrum

Assuming inverted ordering of the mass spectrum

If such symmetry is the discrete quaternion group $Q_8$:

\[
M^{(1)}_\nu = \begin{pmatrix} a & c & d \\ c & 0 & b \\ d & b & 0 \end{pmatrix} \quad M^{(2)}_\nu = \begin{pmatrix} a & c & d \\ c & b & 0 \\ d & 0 & b \end{pmatrix} \quad M^{(3)}_\nu = \begin{pmatrix} 0 & c & d \\ c & a & b \\ d & b & a \end{pmatrix}
\]
A look at data on fermion mixing

- No hierarchies between quark and lepton $\theta_{12}$ and $\theta_{13}$.
- Large disparity in the 2 - 3 sector: $\theta_{q_{23}} \ll \theta_{l_{23}}$.
- In first approximation $\theta_{q_{23}} \approx 0$ and $\theta_{q_{13}} \approx 0$. 
Why to worry about the origin of maximal 2-3 mixing?

\[ \sin^2 2\theta_{23} > 0.94 \iff 38^\circ < \theta_{23} < 52^\circ \ (90\% \text{C.L.}) \]

(no precision measurements in next generation experiments)

- For any possible neutrino mass spectrum is, \( \theta_{23} \sim \pi/4 \) determines the dominant structure of the mass matrix \( M_\nu \) (exception: \( M_\nu \approx I_3 \)).

- \( M_\nu \) structure is stable under radiative corrections \( \Rightarrow \) RGE running from GUT to EW scale cannot generate large \( \theta_{23} \) from small (exception: \( M_\nu \approx I_3 \)).

- \( (\nu_\alpha \ l_\alpha)^T \) is SU(2)\(_L\) isodoublet \( \Rightarrow \) flavor alignment expected between \( \nu_\alpha \) and \( l_\alpha \) : cancellation between mixing in \( M_\nu \) and \( M_l \) (that is the case for quarks: \( \theta_{q23} \approx 2^\circ \)).
**Discrete Family Symmetry**

- Standard Model: 13 free parameters in the Yukawa sector.
- Non-zero Majorana (Dirac) neutrino masses: additional 9 (7) parameters in the neutrino mass matrix.
- Family (or “flavor” or “horizontal”) symmetry: relations among masses and mixing of the 3 fermion generations.
- Discrete groups have more low dimensional representations than continuous Lie groups.
- Features of fermion mixing related to the structure of the EW Higgs sector.
\textbf{QUATERNION GROUPS FOR FLAVOR PHYSICS}

• D.Chang, W.-Y. Keung and G.Senjanovic, PRD 42 (1990) 1599, \textit{Neutrino Magnetic Moment}


Fermion Mass Matrices

• K.S.Babu and J.Kubo, hep-ph/0411226,

SUSY Flavor Model

\textbf{Quaternion numbers}: \(a + i_1 b + i_2 c + i_3 d \in (Q, \cdot )\)

\((i_j)^2 = -1, \quad i_j i_k = \varepsilon_{jkl} i_l : \text{non Abelian!}\)

\(Q_8 = \{\pm 1, \pm i_1, \pm i_2, \pm i_3\} \subset SU(2) :\)

\((\psi_1 \quad \psi_2)^T \rightarrow \pm i \quad \sigma_j \quad (\psi_1 \quad \psi_2)^T\)
Quaternions on hep-ph

- D.Chang, W.-Y. Keung and G.Senjanovic, Neutrino Magnetic Moment, PRD 42 (1990) 1599
- *R.Anderson and G.C.Joshi, History of Quaternions in

**QUATERNION GROUPS FOR FLAVOR PHYSICS**

- D.Chang, W.-Y. Keung and G.Senjanovic, PRD 42 (1990) 1599, Neutrino Magnetic Moment
  P.H.Frampton and O.C.W.Kong, hep-ph/9502395;
- K.S.Babu and J.Kubo, hep-ph/0411226, SUSY Flavor Model
Fermion assignments under $Q_8$

- Irreducible representations:
  \[ 1^{++}, 1^{+-}, 1^{-+}, 1^{--}, 2 \]
  Two parities distinguish the 1-dim irreps.

- The 3 generation of fermions transform as
  \[
  \begin{pmatrix}
  u_i \\
  d_i
  \end{pmatrix},
  u_i^c, d_i^c \sim 1^{--}, 1^{--}, 1^{++}, 1^{++} ;
  \begin{pmatrix}
  \nu_i \\
  l_i
  \end{pmatrix},
  l_i^c \sim 1^{++}, 2.
  \]

- Basic tensor product rule:
  \[ 2 \times 2 = 1^{++} + (1^{--} + 1^{+-} + 1^{-+}) \]
Yukawa coupling structure

- Yukawa couplings: $Y_{ij}^k \Psi_i \psi^c_j \Phi_k$
  The matrix structure depends on $\Phi_k$ assignments.
- Two Higgs doublets: $\Phi_1 \sim 1^{++}$, $\Phi_2 \sim 1^{+-}$

<table>
<thead>
<tr>
<th>Quark sector</th>
<th>Charged lepton sector</th>
</tr>
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<tbody>
<tr>
<td>$M_{u,d} = \begin{pmatrix} av_1 &amp; dv_2 &amp; 0 \ ev_2 &amp; bv_1 &amp; 0 \ 0 &amp; 0 &amp; cv_1 \end{pmatrix}$</td>
<td>$M_l = \begin{pmatrix} av_1 &amp; 0 &amp; 0 \ 0 &amp; cv_2 &amp; bv_1 \ 0 &amp; -bv_1 &amp; -cv_2 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

Only 1-2 Cabibbo mixing

Only 2-3 maximal mixing
The neutrino sector

- **Majorana mass term:** \( \nu_\alpha M_{\alpha\beta} \nu_\beta \)

  \( \nu_\alpha \sim 1^{++}, \ 2 \Rightarrow M_{\alpha\beta} \sim \begin{pmatrix} 1^{++} & 2 \\ \ldots & 1^{--} + 1^{+-} \\ \ldots & \ldots \\ & 1^{--} - 1^{++} \end{pmatrix} \)

- \( M_{\alpha\beta} \) depends on which are the superheavy fields.

  Higgs triplet VEVs \( < \xi_i^0 > \) (**Type II seesaw**):

  \( Y_{\alpha\beta} L_\alpha L_\beta \xi_i + \text{h.c.} \quad u_i = < \xi_i^0 > \sim v^2 / M_\xi \)

- To obtain \( M_{\alpha\beta} \) phenomenologically viable, e.g.:

  \( \xi_1 \sim 1^{++} \quad \xi_2 \sim 1^{+-} \quad (\xi_3, \xi_4) \sim 2 \)

  (in general, \( \xi_1 \) and \( \xi_2 \) in 2 different 1-dim irreps)
Maximal mixing in 2-3 sector

\[ M_l = U_L D_l U_R^\dagger, \quad M_\nu = U_\nu^* D_\nu U_\nu^\dagger, \quad U_{PMNS} = U_L^\dagger U_\nu \]

<table>
<thead>
<tr>
<th>Scalar Fields</th>
<th>( M_1 )</th>
<th>( M_\nu )</th>
</tr>
</thead>
</table>
| \( \phi_1 \sim 1^{+-}, \phi_2 \sim 1^{--} \) \( \xi_1 \sim 1^{--}, \xi_2 \sim 1^{--} \) | \[
\begin{pmatrix}
m_\mu & 0 \\
0 & m_\tau
\end{pmatrix}
\] | \[
\begin{pmatrix}
a & b \\
b & a
\end{pmatrix}
\] |
| \( \phi_1 \sim 1^{++}, \phi_2 \sim 1^{--} \) \( \xi_1 \sim 1^{--}, \xi_2 \sim 1^{--} \) | \[
\begin{pmatrix}
c & d \\
-d & c
\end{pmatrix}
\] | \[
\begin{pmatrix}
m_2 & 0 \\
0 & m_3
\end{pmatrix}
\] |
| \( \phi_1 \sim 1^{++}, \phi_2 \sim 1^{--} \) \( \xi_1 \sim 1^{-+}, \xi_2 \sim 1^{--} \) | \[
\begin{pmatrix}
c & d \\
-d & c
\end{pmatrix}
\] | \[
\begin{pmatrix}
a & b \\
b & a
\end{pmatrix}
\] |
**Q₈ neutrino mass matrices**

- Neutrino sector depends on the choice of superheavy fields. Contributions to $M_\nu$ from scalar triplet VEVs $<\xi_i>$: $Y_{\alpha\beta} L_{\alpha} L_{\beta} \xi_i + \text{h.c.}$

- $\xi_1$ and $\xi_2$ in two different Q₈ 1-dim irreps (remember that doublets are: $\Phi_1 \sim 1^{++}$, $\Phi_2 \sim 1^{+-}$)

- The solar mixing requires $(\xi_3, \xi_4) \sim 2$:

  $$\Delta M_\nu = \begin{pmatrix} 0 & c & d \\ c & 0 & 0 \\ d & 0 & 0 \end{pmatrix}$$
Q$_8$ predictions for neutrinos (I)

Scenarios (1) or (2): $M_{\nu} = \begin{pmatrix} a & c & d \\ c & 0 & b \\ d & b & 0 \end{pmatrix}$ or $\begin{pmatrix} a & c & d \\ c & b & 0 \\ d & 0 & b \end{pmatrix}$

- Two texture zeros or one zero and one equality
- Inverted hierarchy (with $m_3 > 0.015$ eV) or quasi-degeneracy (masses up to present upper bound)
- Atmospheric mixing related to 1-3 mixing: $\theta_{23} = \pi/4 \Leftrightarrow \theta_{13} = 0$
- Observable neutrinoless 2$\beta$-decay: $m_{ee} = a > 0.02$ eV
**Q₈ predictions for neutrinos (II)**

One cannot tell scenario (1) from (2): they are distinguished by the **Majorana phase** between \( m_2 \) and \( m_3 \), which presently cannot be measured!

**Scenario (3):**

\[
M_\nu = \begin{pmatrix}
0 & c & d \\
c & a & b \\
d & b & a
\end{pmatrix}
\]

- One texture **zero** and one **equality**
- **Normal hierarchy**: \( 0.035 \text{ eV} < m_3 < 0.065 \text{ eV} \)
- \( \sin\theta_{13} < 0.2 \Rightarrow \sin^22\theta_{23} > 0.98 \)
- No neutrinoless 2\(\beta\) decay: \( m_{ee} = 0 \)
**Phenomenology of Q₈ Higgs sector**

- 2 Higgs doublets distinguished by a parity:
  \( \Phi_1 \sim 1^{++}, \Phi_2 \sim 1^{+-}, \langle \Phi_i \rangle = v_i \)
- FCNCs in quark 1-2 sector: \( \Delta m_K, \Delta m_D \) at tree level:
  \[
  \frac{\Delta m_K}{m_K} \simeq \frac{B_K f_K^2}{3 m_h^2} \left( \frac{v_1^2 + v_2^2}{v_1^2 v_2^2} \right) s_{L}^2 c_{L}^2 m_d m_s
  \]
  For \( m_h=100 \text{GeV} \), \( \Delta m_K/m_K \sim 10^{-15} \) (exp.: 7 \( 10^{-15} \)), \( \Delta m_D/m_D \sim 10^{-15} \) (exp.: < 2.5 \( 10^{-14} \)).
- No FCNCs in lepton 2-3 sector: maximal mixing implies diagonal couplings to both Higgs doublets.
- The non-standard Higgs \( h^0 \) decays into \( \tau^+ \tau^- \) and \( \mu^+ \mu^- \) with comparable strength (\( \sim m_\tau / m_W \)).
Summary

• Data on **lepton masses and mixing** are nowadays very constraining; the largest mass difference (**2-3 sector**) is associated with the largest mixing!

• **$Q_8$** is the smallest non-trivial SU(2) subgroup; $Q_8$ accommodates in different representations the 3 generations of quarks and leptons.

• **Neutrino $Q_8$ phenomenology:**
  – Accomodates **maximal 2-3 mixing** (and all other data);
  – Constrains **mass spectrum and $m_{ee}$**;
  – Explains **texture zeros or equalities** in the mass matrix.

• **Higgs $Q_8$ phenomenology:**
  – **Two doublets** model with parity symmetry;
  – Non-standard **decay rates into $\tau^+ \tau^-$ and $\mu^+ \mu^-$**.