Optical resonators in space

Claus Lämmerzahl

Centre for Applied Space Technology and Microgravity (ZARM), University of Bremen, 20359 Bremen, Germany

Moriond Meeting, La Thuile, March 11 – 18, 2007
Outline

1. Science with optical resonators
2. Time
1 Science with optical resonators

2 Time

3 The resonator in the gravity gradient
   • The problem
   • Simplified model
   • Basic equation and boundary conditions
   • Analytical solution
   • Comparison between Analytical and FEM Solution
   • Numerical solution for the full problem
   • Thermal gradient
Outline

1 Science with optical resonators

2 Time

3 The resonator in the gravity gradient
   - The problem
   - Simplified model
   - Basic equation and boundary conditions
   - Analytical solution
   - Comparison between Analytical and FEM Solution
   - Numerical solution for the full problem
   - Thermal gradient

4 Summary and Outlook
Outline

1. Science with optical resonators
2. Time
3. The resonator in the gravity gradient
   - The problem
   - Simplified model
   - Basic equation and boundary conditions
   - Analytical solution
   - Comparison between Analytical and FEM Solution
   - Numerical solution for the full problem
   - Thermal gradient
4. Summary and Outlook
Science with optical resonators

e.g. OPTIS
The mission OPTIS aims at improving tests of the *foundations of Special and General Relativity* by up to three orders of magnitude.

<table>
<thead>
<tr>
<th>test</th>
<th>method</th>
<th>present accuracy</th>
<th>OPTIS accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 isotropy of speed of light</td>
<td>cavity–cavity comparison</td>
<td>$10^{-16}$</td>
<td>$10^{-19}$</td>
</tr>
<tr>
<td>2 constancy of speed of light</td>
<td>cavity–clock comparison</td>
<td>$10^{-16}$</td>
<td>$10^{-19}$</td>
</tr>
<tr>
<td>3 time dilation – Doppler effect</td>
<td>cavity–clock comparison</td>
<td>$10^{-19}$ $2 \cdot 10^{-7}$</td>
<td>$10^{-9}$</td>
</tr>
<tr>
<td>4 universality of gravit. redshift</td>
<td>laser link</td>
<td>$1.7 \cdot 10^{-2}$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>5 absolute gravitational redshift</td>
<td>time transfer</td>
<td>$1.4 \cdot 10^{-4}$</td>
<td>$10^{-8}$</td>
</tr>
</tbody>
</table>
Outline

1. Science with optical resonators

2. Time

3. The resonator in the gravity gradient
   - The problem
   - Simplified model
   - Basic equation and boundary conditions
   - Analytical solution
   - Comparison between Analytical and FEM Solution
   - Numerical solution for the full problem
   - Thermal gradient

4. Summary and Outlook
Schwarzschild space–time

Metric

\[ ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2 \]

Clocks

Reading of clocks

\[ s = \int_{\text{orbit}} ds = \int_{\text{orbit}} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} = \int_{\text{orbit}} \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} dt \]
Orbits in Schwarzschild space–time

**Equation of motion**

General

\[ D_v v = 0 \iff \frac{d^2 x^\mu}{ds^2} + \left\{ \frac{\mu}{\rho\sigma} \right\} \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} \]

Conservation laws

\[ E = g_{00} \frac{dt}{ds}, \quad L = r^2 \frac{d\varphi}{ds} \]

Yields three equations

\[
\left( \frac{dr}{d\varphi} \right)^2 = \frac{r^4}{L^2} \left( E^2 - \left( 1 - \frac{2m}{r} \right) \left( \epsilon + \frac{L^2}{r^2} \right) \right) \\
\left( \frac{dr}{ds} \right)^2 = E^2 - \left( 1 - \frac{2m}{r} \right) \left( \epsilon + \frac{L^2}{r^2} \right) \\
\left( \frac{dr}{dt} \right)^2 = \frac{1}{E^2} \left( 1 - \frac{2m}{r} \right)^2 \left( E^2 - \left( 1 - \frac{2m}{r} \right) \left( \epsilon + \frac{L^2}{r^2} \right) \right) .
\]
Orbits in Schwarzschild space–time

Effective potential

Angular momentum
\[ \frac{L^2}{2m^2r^2} \]

Newton potential
\[ \frac{m}{r} \]

Relativistic gravity
\[ \frac{L^2m}{r^3} \]
Orbits in Schwarzschild space–time

Final equations

With substitutions

\[
\begin{align*}
\frac{d\varphi}{2} &= \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}} \\
\frac{ds}{2} &= \lambda L \frac{dx}{(x + \frac{1}{3})^2 \sqrt{4x^3 - g_2 x - g_3}} \\
\frac{dt}{2} &= \lambda LE \frac{dx}{(x + \frac{1}{3})^2 \left(\frac{2}{3} - x\right) \sqrt{4x^3 - g_2 x - g_3}},
\end{align*}
\]

with

\[g_2, g_3 \quad \leftrightarrow \quad L, E, M\]
Orbits in Schwarzschild space–time

**Solution for orbit (Hagihara 1931)**

\[
\begin{align*}
  r(\varphi) &= \frac{2m}{\frac{1}{3} + \wp(\frac{\varphi}{2}; g_2, g_3)} , \\
  r(\varphi) &= \frac{2m}{\frac{1}{3} + \wp(\frac{\varphi}{2} + i\omega_2; g_2, g_3)}
\end{align*}
\]

**Solution for proper time**

\[
\begin{align*}
  s(\varphi) + s_0 &= \frac{L}{2(\mu - 1)} \left( -\zeta(\frac{\varphi}{2} + \varphi_1) - \zeta(\frac{\varphi}{2} - \varphi_1) - \varphi \wp(\varphi_1) \\
  &\quad + \sqrt{\frac{\lambda}{\mu - 1}} \left( \ln \frac{\sigma(\frac{\varphi}{2} + \varphi_1)}{\sigma(\frac{\varphi}{2} - \varphi_1)} - \varphi \zeta(\varphi_1) \right) \right)
\end{align*}
\]

where

\[
\begin{align*}
  \zeta(z) &= \frac{1}{z} - \int^z \left( \frac{1}{u^2} - \wp(u) \right) du , \\
  \sigma(z) &= z \exp \left( \int^z \left( \zeta(u) - \frac{1}{u} \right) \right) , \\
  \frac{d}{dz} \zeta(z) &= -\wp(z) , \\
  \frac{d}{dz} \ln \sigma(z) &= \zeta(z)
\end{align*}
\]
Orbits in Schwarzschild space–time

Measured frequency

$$\frac{dt}{ds} = \frac{1}{\frac{2}{3} - \varphi \left( \frac{\varphi}{2} ; g_2, g_3 \right)}$$

$$\frac{dt}{ds} = \frac{1}{\frac{2}{3} - \varphi \left( \frac{\varphi}{2} + i\omega_2 ; g_2, g_3 \right)}$$
Proper time in Schwarzschild space-time

quasi-elliptic orbit
Proper time in Schwarzschild space-time

quasi-hyperbolic orbit
Proper time in Schwarzschild space-time

quasi–hyperbolic orbit

C. Lämmerzahl (ZARM, Bremen)
Proper time in Schwarzschild space-time

quasi-parabolic orbit
Proper time in Schwarzschild space-time

Clock reading

- Exact trajectories
- Exact proper time
- Exact light trajectories
- Exact frequency reading (gravitational redshift) during orbit

To be compared with Teyassandier, Linet, Blanchet, Wolf, .....: calculations in 2nd PN order
Inclusion of higher gravitational multipoles
Outline

1 Science with optical resonators

2 Time

3 The resonator in the gravity gradient
   - The problem
   - Simplified model
   - Basic equation and boundary conditions
   - Analytical solution
   - Comparison between Analytical and FEM Solution
   - Numerical solution for the full problem
   - Thermal gradient

4 Summary and Outlook
Optical resonators in space

A. Peters
The resonator in the gravity gradient

Optical resonators

A. Peters
C. Lämerzahl (ZARM, Bremen)
Optical resonators
The resonator in the gravity gradient

The problem

Simplified model

Basic equation and boundary conditions

Analytical solution

Comparison between Analytical and FEM Solution

Numerical solution for the full problem

Thermal gradient

Outline

1. Science with optical resonators

2. Time

3. The resonator in the gravity gradient
   - The problem
   - Simplified model
   - Basic equation and boundary conditions
   - Analytical solution
   - Comparison between Analytical and FEM Solution
   - Numerical solution for the full problem
   - Thermal gradient

4. Summary and Outlook
The problem

Gravity gradient

- Tidal force = difference of acceleration over extension of body

\[ \Delta a = \frac{\partial a}{\partial r} L = \frac{\partial^2 U}{\partial r^2} L \]

- Cannot be transformed away, cannot be eliminated

Rough estimate of expected effect

- Typically \( \Delta a \approx 2 \cdot 10^{-8} \text{ m/s}^2 \) for \( L = 5 \text{ cm} \)

- Assumption: this \( \Delta a \) acts on the top surface of resonator. Hooke’s law of elasticity

\[ \frac{\Delta L}{L} = \frac{1}{E} \frac{F}{A} = \frac{\rho L}{E} \Delta a \approx 10^{-17} \]

for \( F = m \Delta a = \rho L^3 \Delta a \), assuming values for Zerodur.

- Larger than OPTIS science goal

⇒ Has to be subtracted from signal
Outline

1. Science with optical resonators

2. Time

3. The resonator in the gravity gradient
   - The problem
   - Simplified model
   - Basic equation and boundary conditions
   - Analytical solution
   - Comparison between Analytical and FEM Solution
   - Numerical solution for the full problem
   - Thermal gradient

4. Summary and Outlook
Deforming force in body coordinates \((r, z)\)

\[
\mathbf{K} = \rho \nabla \left( \frac{GM}{r_{\text{com}}} \left( r^2 - 2z^2 \right) \right),
\]

\(G =\) gravitational constant, \(M =\) mass of the Earth
Outline

1. Science with optical resonators

2. Time

3. The resonator in the gravity gradient
   - The problem
   - Simplified model
   - Basic equation and boundary conditions
     - Analytical solution
     - Comparison between Analytical and FEM Solution
     - Numerical solution for the full problem
     - Thermal gradient

4. Summary and Outlook
Basic Equation

Basic equilibrium equation \((K = \text{volume force, gravitaty gradient})\)

\[
D_j \sigma^j_i + K_i = 0
\]

Deformation tensor

\[
\epsilon = \frac{1}{2} \mathcal{L}_{ug}
\]

Hooke's law

\[
\sigma^{ij} = c^{ijkl} \epsilon_{kl}
\]

Homogenous isotropic elastic solid

\[
c^{ijkl} = \mu (g^{ik} g^{jl} + g^{jk} g^{il}) + \lambda g^{ij} g^{kl}
\]

Lamé-Navier equation \((\Delta = g^{kl} D_k D_l, (\text{grad div } u)^i = g^{il} D_l D_n u^n)\)

\[
\mu \Delta u^i + (\lambda + \mu) (\text{grad div } u)^i + K^i = 0
\]

Homogenous part \((K = 0)\) implies the biharmonic equation

\[
\Delta \Delta u = 0
\]
Boundary Conditions

Typical boundary conditions

- $u^i(x_0) = u_{i0}$, a prescribed initial value
- $\sigma_{ij}n_j = F_i$, $F$ = surface force (pressure)

Our boundary conditions

- The pressure $p$ is zero and therefore we have the second condition with the surface force $F = 0$. 
Outline

1. Science with optical resonators
2. Time
3. The resonator in the gravity gradient
   - The problem
   - Simplified model
   - Basic equation and boundary conditions
   - Analytical solution
     - Comparison between Analytical and FEM Solution
     - Numerical solution for the full problem
     - Thermal gradient
4. Summary and Outlook
Particular and homogeneous solution

- We search for a solution $u$ as sum of a particular and a homogenous solution:
  \[ u = u_p + u_h \]
- The particular solution is chosen independently of the boundary conditions.
- The boundary conditions with respect to the homogenous equation has to be reformulated.
- The negative boundary values of the particular solution gives the boundary conditions for the homogenous equation:
  \[ f^i := \sigma^i_{h} n_j = -\sigma^i_{p} n_j \]
Particular solution

Ansatz (Lurie 1970)

\[ u = \nabla \psi \]

where \( \psi \) is a scalar function. This gives a particular solution of the axially symmetric basic equation in the form

\[ u^r(r, z) = a \left( -\frac{1}{4} r^3 + 2cr \right) \]

\[ u^z(r, z) = a \left( \frac{2}{3} z^3 + 2dz \right) \]

where \( a = \frac{1 - 2\nu}{2\mu(1 - 2\nu)} \rho \frac{GM}{r_{com}^3} \), \( c \) and \( d \) arbitrary constants.
Homogeneous boundary conditions

The boundary conditions of the homogenous equation are

\[ f^r(R, z) = \sigma^{rj} n_j(R, z) = a_2 z^2 + a_0 \]
\[ f^z(R, z) = \sigma^{zj} n_j(R, z) = 0 \]
\[ f^r(r, \pm L) = \sigma^{rj} n_j(r, \pm L) = 0 \]
\[ f^z(r, \pm L) = \sigma^{zj} n_j(r, \pm L) = b_2 r^2 + b_0 \]

with some constants \(a_2, a_0, b_2, b_0\). It is not possible to obtain linear boundary conditions.
Homogeneous solution

Love–ansatz in terms of Love function $\chi$

\[ u^h_r = -\frac{1 + \nu}{E} \frac{\partial^2 \chi}{\partial r \partial z} \]
\[ u^h_z = \frac{1 + \nu}{E} \left( (1 - 2\nu) \nabla^2 \chi + \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r} \frac{\partial \chi}{\partial r} \right) \]

stress tensor

\[ \sigma^h_{rr} = \frac{\partial}{\partial z} \left( \nu \nabla^2 \chi - \frac{\partial^2 \chi}{\partial r^2} \right) \]
\[ \sigma^h_{rz} = \frac{\partial}{\partial r} \left( (1 - \nu) \nabla^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right) \]
\[ \sigma^h_{zz} = \frac{\partial}{\partial z} \left( (2 - \nu) \nabla^2 \chi - \frac{\partial^2 \chi}{\partial z^2} \right) \]
\[ \sigma^h_{\phi\phi} = \frac{\partial}{\partial r} \left( \nu \nabla^2 \chi - \frac{1}{r} \frac{\partial \chi}{\partial r} \right) . \]

The Love function $\chi$ necessarily fulfills the biharmonic equation

\[ \nabla^2 \nabla^2 \chi = 0 . \]
Homogeneous solution

Ansatz for Love function guided by symmetry of problem (Papkovitch-Neuber approach, Meleshko)

\[ \chi = B_0 z^3 + \sum_{j=1}^{\infty} \left( A_j \frac{\sinh(\lambda_j z)}{\sinh(\lambda_j L)} + B_j z \frac{\cosh(\lambda_j z)}{\sinh(\lambda_j L)} \right) \frac{J_0(\lambda_j r)}{\lambda_j^2} + D_0 r^2 z + \sum_{n=1}^{\infty} \left( C_n \frac{I_0(k_n r)}{I_1(k_n R)} + D_n r \frac{I_1(k_n r)}{I_1(k_n R)} \right) \frac{\sin(k_n z)}{k_n^2}. \]

\( J_0 \) Bessel functions of first kind and order zero, \( I_1 \) are the modified Bessel functions of order one.
\( \zeta_j = \lambda_j R \) are zeros of Bessel functions \( J_1(\zeta_j) = 0 \), and \( k_n = \frac{n\pi}{L} \) where \( n \) is an integer number.

Ansatz in boundary conditions \( \rightarrow \) determination of coefficients
The constants are determined by

\[ A_j = -\frac{B_j}{\lambda_j} \left( 2\nu + \lambda_j L \coth(\lambda_j L) \right) \]

\[ C_n = -\frac{D_n}{k_n} \left( (2-2\nu) + k_n R \frac{I_0(k_n R)}{I_1(k_n R)} \right) \]

\[ D_n = -\frac{1}{k_n R \left( \frac{I_0(k_n R)^2}{I_1(k_n R)^2} - 1 \right) - \frac{2-2\nu}{k_n R}} \left( \frac{4\hat{c}_2(-1)^n}{k_n^2} + \sum_{j=1}^{\infty} B_j (-1)^n \frac{4\lambda_j k_n^2}{L(k_n^2 + \lambda_j^2)^2} J_0(\lambda_j R) \right) \]

\[ B_j = \frac{1}{J_0(\lambda_j R) \left( \coth(\lambda_j L) + \frac{\lambda_j L}{\sinh^2(\lambda_j L)} \right)} \left( \frac{2}{\lambda_j^2} \hat{c}_2 - \sum_{n=1}^{\infty} (-1)^n D_n k_n \left( \frac{4\lambda_j^2}{R(k_n^2 + \lambda_j^2)^2} \right) \right) \]

\[ B_0 = \frac{1}{6} \left( -\hat{c}_1 + \hat{c}_2 \left( \frac{R^2}{4} - L^2 \right) - S \right) - D_0 \]

\[ D_0 = \frac{1}{2(1+\nu)} \left( (1-\nu) (\hat{c}_1 + \hat{c}_2 L^2 + S) + \nu \hat{c}_2 \frac{R^2}{4} - Z \right) \]
Homogeneous solution

with

\[
\hat{c}_0 = \gamma \left( \frac{4c\nu}{1 - \nu} + 2(L^2 + d) \right) \quad \hat{c}_1 = \gamma \left( \frac{(2\nu - 3)}{4(1 - \nu)} R^2 + \frac{2(c + \nu d)}{1 - \nu} \right) \quad \hat{c}_2 = \gamma \frac{2\nu}{1 - \nu}
\]

and

\[
S = R + Z
\]

\[
R = \sum_{j=1}^{\infty} B_j \left( \frac{(1 - \lambda_j L \coth(\lambda_j L))}{\sinh(\lambda_j L)} \right) J_0(\lambda_j R) + \sum_{n=1}^{\infty} D_n \left( k_n R \left( \frac{I_0(k_n R)^2}{I_1(k_n R)^2} - 1 \right) - \frac{2 - 2\nu}{k_n R} \right)
\]

\[
Z = \sum_{j=1}^{\infty} B_j \left( \coth(\lambda_j L) + \frac{\lambda_j L}{\sinh^2(\lambda_j L)} \right) + \sum_{n=1}^{\infty} (-1)^n D_n \left( 2 - k_n R \frac{I_0(k_n R)}{I_1(k_n R)} \right) \frac{1}{I_1(k_n R)}
\]

solution forms an infinite system of equations whose convergence can be shown. So it can be approximately solved by reducing it to a finite system, that is, by expanding the sums only to \( n = N \) and \( j = J \). Then we have a system of \( N + J \) equations. By increasing the values of \( N \) and \( J \) one can improve the accuracy of the solution and find their limits.
Table 1 shows the results from numerical evaluation of the infinite system of equations for different orders $N = J$. Although the infinite series converge very quickly, an expansion to higher orders $N = J$ still gives an improvement of accuracy.

**Table:** Comparison of the analytical solution at point $r = 1 = R$, $z = 2 = L$ for different expansion orders $N = J$ of the infinite sums in equation system above.

<table>
<thead>
<tr>
<th>$N = J$</th>
<th>$\xi_r$</th>
<th>$\xi_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>$-6.471 \cdot 10^{-15}$</td>
<td>$1.637 \cdot 10^{-13}$</td>
</tr>
<tr>
<td>1000</td>
<td>$-6.233 \cdot 10^{-15}$</td>
<td>$1.629 \cdot 10^{-13}$</td>
</tr>
<tr>
<td>1700</td>
<td>$-6.218 \cdot 10^{-15}$</td>
<td>$1.628 \cdot 10^{-13}$</td>
</tr>
</tbody>
</table>

First analytical solution for gravity gradient volume force (S. Scheithauer & C.L., *Class. Quantumm Grav.* 2006)
The resonator in the gravity gradient

Analytical Solution

Displacement field. The infinite sums in the analytical solution were expanded to $N = J = 1700$.

FEM solution. Deformation scaled by a factor of $6 \cdot 10^{13}$. Right: Deformed cylinder shape and original finite element mesh. Left: the scale shows the $z$ displacements.
Outline

1. Science with optical resonators

2. Time

3. The resonator in the gravity gradient
   - The problem
   - Simplified model
   - Basic equation and boundary conditions
   - Analytical solution
   - Comparison between Analytical and FEM Solution
   - Numerical solution for the full problem
   - Thermal gradient

4. Summary and Outlook
Comparison between Analytical and FEM Solution

- Commercial FEM code ANSYS.
- Cylinder divided into approximately 110,000 hexahedron elements
- Structured finite element mesh

Table: Comparison between analytical and FEM solution. Displacements $\xi_r$ and $\xi_z$. The cylinder boundaries are $r = R = 1$ and $z = \pm L = \pm 2$. In analytical solution $N = J = 1700$.

<table>
<thead>
<tr>
<th></th>
<th>r</th>
<th>z</th>
<th>$\xi_r$</th>
<th>$\xi_z$</th>
<th>Analytical</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>$-6.218 \cdot 10^{-15}$</td>
<td>$1.628 \cdot 10^{-13}$</td>
<td>$-6.202 \cdot 10^{-15}$</td>
<td>$1.625 \cdot 10^{-13}$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0</td>
<td>$-3.491 \cdot 10^{-14}$</td>
<td>0</td>
<td>$-3.486 \cdot 10^{-14}$</td>
<td>$-1.66 \cdot 10^{-23}$</td>
</tr>
<tr>
<td></td>
<td>0.49507$\sqrt{2}$</td>
<td>1</td>
<td>$-2.059 \cdot 10^{-14}$</td>
<td>$1.151 \cdot 10^{-13}$</td>
<td>$-2.062 \cdot 10^{-14}$</td>
<td>$1.150 \cdot 10^{-13}$</td>
</tr>
<tr>
<td></td>
<td>0.11888$\sqrt{2}$</td>
<td>1</td>
<td>$-5.208 \cdot 10^{-15}$</td>
<td>$1.183 \cdot 10^{-13}$</td>
<td>$-5.236 \cdot 10^{-15}$</td>
<td>$1.182 \cdot 10^{-13}$</td>
</tr>
</tbody>
</table>

Small differences

Based on the nature of the FEM analysis:
- Elements are 'finite'
- During FEM analysis at least three points must be fixed in order to prevent rigid rotations.
Outline

1 Science with optical resonators

2 Time

3 The resonator in the gravity gradient
   • The problem
   • Simplified model
   • Basic equation and boundary conditions
   • Analytical solution
   • Comparison between Analytical and FEM Solution
   • Numerical solution for the full problem
   • Thermal gradient

4 Summary and Outlook
Numerics

Structural analysis within the FEM program ANSYS. OPTIS resonator = cube with side length 6 cm having three orthogonal drillings; drillings are closed by circular mirrors at each end. Material = Corning ULE (Ultra Low Expansion) glass, that means elasticity modulus $E = 67.6$ GPa, Possion number $\nu = 0.17$, density $\rho = 2210$ kg/m$^3$.

Figure: Deformation of the OPTIS resonator under the influence of a gravity gradient force. The color bar gives the values of the $z$-displacements in meter.
FEM analysis: rotating resonator

OPTIS resonator with gravity gradient at point $\phi_y = \pi / 2$: Mirrors on the $x$-axis.

Left side: Displacements at the mirror midpoints, dotted line = mirror on negative original coordinate point, full line = mirror on positive original coordinate point.

Right side: relative displacements between two opposing mirrors.
FEM analysis: rotating resonator

OPTIS resonator with gravity gradient at point $\phi_y = \pi/2$ on its orbit around the Earth: Mirrors on the $y$-axis, Left side: Displacements at the mirror midpoints, dotted line = mirror on negative original coordinate point, full line = mirror on positive original coordinate point; Right side: relative displacements between two opposing mirrors.
OPTIS resonator with gravity gradient at point $\phi_y = \pi/2$ on its orbit around the Earth: Mirrors on the $z$-axis, Left side: Displacements at the mirror midpoints, dotted line = mirror on negative original coordinate point, full line = mirror on positive original coordinate point; Right side: relative displacements between two opposing mirrors
OPTIS resonator with gravity gradient: Contour plots of the relative mirror displacements $dx$ on the $x$-axis. The data corresponds to an orbital rotation around the inertial $y$ axis for half an orbit with angle $\phi_y$ and a rotation of the resonator around its body $z$-axis with angle $\phi_z$. 
OPTIS resonator with gravity gradient: Contour plots of the relative mirror displacements $dy$ on the $x$-axis. The data corresponds to an orbital rotation around the inertial $y$ axis for half an orbit with angle $\phi_y$ and a rotation of the resonator around its body $z$-axis with angle $\phi_z$. 
OPTIS resonator with gravity gradient: Contour plots of the relative mirror displacements $dz$ on the $x$-axis. The data corresponds to an orbital rotation around the inertial $y$ axis for half an orbit with angle $\phi_y$ and a rotation of the resonator around its body $z$-axis with angle $\phi_z$. 
OPTIS resonator with gravity gradient: Contour plots of the relative mirror displacements $dx$ on the $y$-axis. The data corresponds to an orbital rotation around the inertial $y$ axis for half an orbit with angle $\phi_y$ and a rotation of the resonator around its body $z$-axis with angle $\phi_z$. 
FEM analysis: rotating resonator

OPTIS resonator with gravity gradient: Contour plots of the relative mirror displacements $dy$ on the $y$-axis. The data corresponds to an orbital rotation around the inertial $y$ axis for half an orbit with angle $\phi_y$ and a rotation of the resonator around its body $z$-axis with angle $\phi_z$. 
OPTIS resonator with gravity gradient: Contour plots of the relative mirror displacements $dz$ on the $y$-axis. The data corresponds to an orbital rotation around the inertial $y$ axis for half an orbit with angle $\phi_y$ and a rotation of the resonator around its body $z$-axis with angle $\phi_z$. 
OPTIS resonator with gravity gradient: Contour plots of the relative mirror displacements $dx$ on the $z$-axis. The data corresponds to an orbital rotation around the inertial $y$ axis for half an orbit with angle $\phi_y$ and a rotation of the resonator around its body $z$-axis with angle $\phi_z$. 
OPTIS resonator with gravity gradient: Contour plots of the relative mirror displacements $dy$ on the $z$-axis. The data corresponds to an orbital rotation around the inertial $y$ axis for half an orbit with angle $\phi_y$ and a rotation of the resonator around its body $z$-axis with angle $\phi_z$. 

C. Lämmerzahl (ZARM, Bremen)
OPTIS resonator with gravity gradient: Contour plots of the relative mirror displacements $dz$ on the $z$-axis. The data corresponds to an orbital rotation around the inertial $y$ axis for half an orbit with angle $\phi_y$ and a rotation of the resonator around its body $z$-axis with angle $\phi_z$. 
Outline

1 Science with optical resonators

2 Time

3 The resonator in the gravity gradient
   - The problem
   - Simplified model
   - Basic equation and boundary conditions
   - Analytical solution
   - Comparison between Analytical and FEM Solution
   - Numerical solution for the full problem
   - Thermal gradient

4 Summary and Outlook
deformation of resonator for a thermal gradient of $10^{-9}$ K between upper and lower surface in $z$-direction for Corning ULE glass (thermal expansion coefficient of $\alpha \approx 10^{-9}$ K$^{-1}$).

- Relative deformations between mirrors are $\sim 2.99 \cdot 10^{-20}$ m.
- Zerodur ($\alpha \approx 10^{-7}$ K$^{-1}$): deformations are larger by two orders of magnitude.
- ULE: temperature gradient must be smaller than $10^{-9}$ K over $L = 6$ cm.
- Temperature stability between endpoints of resonator must be better than $10^{-9}$ K on a time scale $L^2/\chi_T$ ($\chi_T =$ temperature conductivity).
Outline

1. Science with optical resonators

2. Time

3. The resonator in the gravity gradient
   - The problem
   - Simplified model
   - Basic equation and boundary conditions
   - Analytical solution
   - Comparison between Analytical and FEM Solution
   - Numerical solution for the full problem
   - Thermal gradient

4. Summary and Outlook