Coherent Bayesian analysis of inspiral signals

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Overview:

1. The Bayesian approach
2. MCMC methods
3. The inspiral signal
4. Priors
5. Example application
The Bayesian approach

- idea: assign probabilities to parameters $\theta$
- pre-experimental knowledge: prior probabilities / -distribution $p(\theta)$
- data model: likelihood $p(y|\theta)$
- application of Bayes’ theorem yields the posterior distribution
  $$p(\theta|y) \propto p(\theta) p(y|\theta)$$
  conditional on the observed data $y$.
- posterior distribution combines the information in the data with the prior information
MCMC methods - what they do

• Problem -
  \textbf{given}: posterior distribution $p(\theta | y)$ (density, function of $\theta$)
  \textbf{wanted}: mode(s), integrals,...

• what MCMC does:
  simulate random draws from (any) distribution, allowing to approximate
  any integral by sample statistic (e.g. means by averages etc.)

• Monte Carlo integration
MCMC methods - how they work

- **Markov Chain Monte Carlo**
- random walk
- **Markov property**: each step in random walk only depends on previous
- **stationary distribution** is equal to the desired posterior $p(\theta|y)$
- most famous: **Metropolis- (Hastings-) sampler**
  especially convenient: normalising constant factors to $p(\theta|y)$ don’t need to be known.
MCMC methods

• Metropolis-algorithm may also be seen as **optimisation algorithm**: improving steps always accepted, worsening steps sometimes (→ *Simulated Annealing*)

• in fact: purpose often *both finding mode(s) and sampling* from them
The inspiral signal

- measurement: time series (signal + noise) at, say, 3 separate interferometers

- **signal**: chirp waveform; 2.5PN amplitude, 3.5PN phase\textsuperscript{1,2}

- **9 parameters**: masses \((m_1, m_2)\), coalescence time \((t_c)\), coalescence phase \((\phi_0)\), luminosity distance \((d_L)\), inclination angle \((\iota)\), sky location \((\delta, \alpha)\) and polarisation \((\psi)\)

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\textsuperscript{1}K.G. Arun et al.: *The 2.5PN gravitational wave polarizations from inspiralling compact binaries in circular orbits*, Class. Quantum Grav. 21, 3771 (2004).

The signal at different interferometers

- **‘local’ parameters** at interferometer $I$:
  
  - sky location $(\delta, \alpha) \rightarrow$ altitude ($\vartheta^{(I)}$) / azimuth ($\varphi^{(I)}$)
  - coalescence time ($t_c$) $\rightarrow$ local coalescence time ($t_c^{(I)}$)
  - polarisation ($\psi$) $\rightarrow$ local polarisation ($\psi^{(I)}$)

- **noise** assumed **gaussian, coloured**; interferometer-specific spectrum

- **likelihood** computation based on Fourier transforms of data and signal

- **noise independent** between interferometers
  $\Rightarrow$ coherent network likelihood is **product** of individual ones
Prior information about parameters

- different locations / orientations equally likely
- masses: uniform across $[1 \, M_\odot, \, 10 \, M_\odot]$
- events spread uniformly across space: $P(d_L \leq x) \propto x^3$
- but: certain SNR required for detection

- cheap SNR substitute: signal amplitude $A$
- primarily dependent on masses, distance, inclination: $A(m_1, m_2, d_L, \iota)$
- introduce sigmoid function linking amplitude to detection probability\textsuperscript{3}

\textsuperscript{3}R. Umstätter et al.: Setting upper limits from LIGO on gravitational waves from SN1987a. Poster presentation; also: paper in preparation.
Resulting (marginal) prior density

\[ \text{total mass } (m_t = m_1 + m_2) \]

\[ \text{luminosity distance } (d_L) \]
Marginal prior density

\[ \text{luminosity distance (d_L)} \]

\[ \text{inclination angle (i)} \]

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Marginal prior densities

individual masses \((m_1, m_2)\)

\[
\begin{array}{cccccc}
2 & 4 & 6 & 8 & 10 \\
\end{array}
\]

(sun masses)

inclination angle \((\iota)\)

\[
\begin{array}{cccccc}
0 & \pi/2 & \pi \\
\end{array}
\]

(radian)
Prior

• prior ‘considers’ **Malmquist effect**

• more realistic settings once **detection pipeline** is set up
MCMC details

- **Reparametrisation**, most importantly: chirp mass $m_c$, mass ratio $\eta$

- **Parallel Tempering**\(^4\)
  several *tempered* MCMC chains running in parallel
  sampling from $p(\theta | y)^{\frac{1}{T_i}}$ for ‘temperatures’ $1 = T_1 \leq T_2 \leq \ldots$

- **Evolutionary MCMC**\(^5\)
  ‘recombination’ steps between chains—motivated by Genetic algorithms

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Example application

- simulated data:
  2 M☉ - 5 M☉ inspiral at 30 Mpc distance
  measurements from 3 interferometers:

<table>
<thead>
<tr>
<th></th>
<th>SNR</th>
</tr>
</thead>
<tbody>
<tr>
<td>LHO (Hanford)</td>
<td>8.4</td>
</tr>
<tr>
<td>LLO (Livingston)</td>
<td>10.9</td>
</tr>
<tr>
<td>Virgo (Pisa)</td>
<td>6.4</td>
</tr>
<tr>
<td>network</td>
<td>15.2</td>
</tr>
</tbody>
</table>

- data: 10 seconds (LHO/LLO), 20 seconds (Virgo) before coalescence, noise as expected at design sensitivities

- computation speed: 1–2 likelihoods / second
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chirp mass ($m_c$)

mass ratio ($\eta$)

individual masses ($m_1$, $m_2$)

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chirp mass ($m_c$) vs mass ratio ($\eta$):

<table>
<thead>
<tr>
<th>Chirp Mass ($m_c$)</th>
<th>Mass Ratio ($\eta$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.685</td>
<td>0.19</td>
</tr>
<tr>
<td>2.690</td>
<td>0.20</td>
</tr>
<tr>
<td>2.695</td>
<td>0.21</td>
</tr>
<tr>
<td>2.700</td>
<td>0.22</td>
</tr>
<tr>
<td>2.705</td>
<td>0.23</td>
</tr>
<tr>
<td>2.710</td>
<td></td>
</tr>
<tr>
<td>2.715</td>
<td></td>
</tr>
</tbody>
</table>

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### some posterior key figures

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mean</th>
<th>95% C.I.</th>
<th>True</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chirp mass ($m_c$)</td>
<td>2.699</td>
<td>(2.692, 2.707)</td>
<td>2.698</td>
<td>$M_\odot$</td>
</tr>
<tr>
<td>Mass ratio ($\eta$)</td>
<td>0.207</td>
<td>(0.192, 0.225)</td>
<td>0.204</td>
<td></td>
</tr>
<tr>
<td>Coalescence time ($t_c$)</td>
<td>12.3455</td>
<td>(12.3421, 12.3490)</td>
<td>12.3450</td>
<td>s</td>
</tr>
<tr>
<td>Luminosity distance ($d_L$)</td>
<td>31.4</td>
<td>(17.4, 43.5)</td>
<td>30.0</td>
<td>Mpc</td>
</tr>
<tr>
<td>Inclination angle ($\iota$)</td>
<td>0.726</td>
<td>(0.159, 1.456)</td>
<td>0.700</td>
<td>rad</td>
</tr>
<tr>
<td>Declination ($\delta$)</td>
<td>-0.498</td>
<td>(-0.539, -0.456)</td>
<td>-0.506</td>
<td>rad</td>
</tr>
<tr>
<td>Right ascension ($\alpha$)</td>
<td>4.657</td>
<td>(4.632, 4.688)</td>
<td>4.647</td>
<td>rad</td>
</tr>
<tr>
<td>Coalescence phase ($\phi_0$)</td>
<td></td>
<td></td>
<td>2.0</td>
<td>rad</td>
</tr>
<tr>
<td>Polarisation ($\psi$)</td>
<td></td>
<td></td>
<td>1.0</td>
<td>rad</td>
</tr>
</tbody>
</table>
MCMC chain 2 — temperature = 2

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MCMC chain 3 — temperature = 4

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MCMC chain 4 — temperature = 8

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Six tempered chains ‘in action’

\[ \log(p(\theta|y)) \]

iteration

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Outlook

- incorporation into a ‘loose net’ detection pipeline for large mass ratio inspirals
- use information supplied by detection pipeline
- further parameters, e.g. spin effects