After reviewing the modified Newtonian dynamics (MOND) proposal, we advocate that the associated phenomenology may actually not result from a modification of Newtonian gravity, but from a mechanism of “gravitational polarization” of some dipolar medium playing the role of dark matter. We then build a relativistic model within standard general relativity to describe (at some phenomenological level) the dipolar dark matter polarizable in a gravitational field. The model naturally involves a cosmological constant, and is shown to reduce to the concordance cosmological scenario (Λ-CDM) at early cosmological times. From the mechanism of gravitational polarization, we recover the phenomenology of MOND in a typical galaxy at low redshift. Furthermore, we show that the cosmological constant Λ scales like \( a_0^2 \), where \( a_0 \) is the constant MOND acceleration scale, in good agreement with observations.

1 Introduction

The mysteries of the nature of dark matter and dark energy are perhaps the most important ones of contemporary cosmology. Dark matter, which accounts for the observed discrepancy between the dynamical and luminous masses of bounded astrophysical systems, is usually formulated within the so-called particle dark matter approach, in which the dark matter consists of unknown non-baryonic particles, e.g. neutralinos as predicted by super-symmetric extensions of the standard model of particle physics (see 1 for a review). Furthermore, the dark matter triggers the formation of large-scale structures by gravitational collapse and explains the distribution of baryonic matter from galaxy cluster scales up to cosmological scales by the non-linear growth of initial perturbations. Simulations suggest some universal dark matter density profile around distributions of ordinary baryonic matter. An important characteristic of dark matter, required by the necessity of clustering matter on small scales, is that it should be cold (or non-relativistic) at the epoch of galaxy formation. Together with the observational evidence of dark energy (presumably a cosmological constant Λ) measured from the Hubble diagram of supernovas, the particle dark matter hypothesis has yielded the successful concordance model of cosmology called Λ-CDM, which reproduces extremely well the observed cosmic microwave background spectrum.

However, despite these successes at cosmological scales, the particle dark matter approach has some difficulties at explaining in a natural way the flat rotation curves of galaxies, one of the most persuasive evidence for the existence of dark matter, and the Tully-Fisher empirical relation between the observed luminosity and the asymptotic rotation velocity of spiral galaxies. In order to deal with these difficulties, all linked with the properties of dark matter at galactic scales, an alternative paradigm has emerged in the name of the modified Newtonian dynamics...
(MOND)\textsuperscript{5,6,7}. Although MOND in its original formulation cannot be considered as a viable physical model, it is now generally admitted that it does capture in a very simple and powerful “phenomenological recipe” a large number of observational facts, that any pertinent model of dark matter should explain.

It is frustrating that the two alternatives \( \Lambda \)-CDM and MOND, which are successful in complementary domains of validity (say the cosmological scale for \( \Lambda \)-CDM and the galactic scale for MOND), seem to be fundamentally incompatible. In the present paper, we shall propose a different approach, together with a new interpretation of the phenomenology of MOND, which has the potential of bringing together \( \Lambda \)-CDM and MOND into a single unifying relativistic model for dark matter and dark energy. This relativistic model will be shown to benefit from both the successes of \( \Lambda \)-CDM at cosmological scales, and MOND at galactic scales.

2 The modified Newtonian dynamics (MOND)

The original idea behind MOND\textsuperscript{5,6,7} is that there is no dark matter, and we witness a violation of the fundamental law of gravity (or of inertia). MOND is designed to account for the basic features of galactic dark matter halos. It states that the “true” gravitational field \( g \) experienced by ordinary matter, say a test particle whose acceleration would thus be \( a = g \), is not the Newtonian gravitational field \( g_\text{N} \), but is actually related to it by

\[
\mu \left( \frac{g}{a_0} \right) g = g_\text{N}.
\]

Here \( \mu \) is a function of the dimensionless ratio \( g/a_0 \) between the norm of the gravitational field \( g = |g| \), and the constant MOND acceleration scale \( a_0 \simeq 1.2 \times 10^{-10} \text{ m/s}^2 \), whose numerical value is chosen to fit the data. The specific MOND regime corresponds to the weak gravity limit, much weaker than the scale \( a_0 \). In this regime (where formally \( g \to 0 \)) we have\textsuperscript{5,6,7}

\[
\mu \left( \frac{g}{a_0} \right) = \frac{g}{a_0} + \mathcal{O}(g^2).
\]

On the other hand, when \( g \) is much larger than \( a_0 \) (formally \( g \to +\infty \)) the usual Newtonian law is recovered, i.e. \( \mu \to 1 \). Various functions \( \mu \) interpolating between the MOND regime and the Newtonian limit are possible, but most of them appear to be rather \textit{ad hoc}. Taken for granted, the MOND “recipe” (1)–(2) beautifully predicts a Tully-Fisher relation and is very successful at fitting the detailed shape of rotation curves of galaxies from the observed distribution of stars and gas (see\textsuperscript{8,9} for reviews). So MOND appears to be more than a simple recipe and may well be related to some new fundamental physics. In any case the agreement of (1)–(2) with a large number of observations calls for a clear physical explanation.

Taking the divergence of both sides of (1), and using the usual Poisson equation for the Newtonian field \( g_\text{N} \), we obtain a local formulation of MOND in the form of the modified Poisson equation\textsuperscript{10}

\[
\nabla \cdot (\mu g) = -4\pi G \rho_\text{b},
\]

where \( \rho_\text{b} \) is the density of baryonic matter. This equation can be derived from a Lagrangian, and that Lagrangian has been the starting point for constructing relativistic extensions of MOND. Such extensions postulate the existence of extra (supposedly fundamental) fields associated with gravity besides the spin-2 field of general relativity. Promoting the Newtonian potential \( U \) to a scalar field \( \phi \), scalar-tensor theories for MOND have been constructed\textsuperscript{11} but shown to be nonviable; essentially because light signals do not feel the presence of the scalar field, since the physical (Jordan-frame) metric is conformally related to the Einstein-frame metric, and the
Maxwell equations are conformally invariant. This is contrary to observations: huge amounts of dark matter are indeed observed by gravitational (weak and strong) lensing.

Relativistic extensions of MOND that pass the problem of light deflection by galaxy clusters have been shown to require the existence of a time-like vector field. The prototype of such theories is the tensor-vector-scalar (TeVeS) theory\textsuperscript{12,13,14}, whose non-relativistic limit reproduces MOND, and which has been extensively investigated in cosmology and at the intermediate scale of galaxy clusters\textsuperscript{15}. Modified gravity theories such as TeVeS have evolved recently toward Einstein-æther like theories\textsuperscript{16,17,18}. Still, recovering the level of agreement of the Λ-CDM scenario with observations at cosmological scales remains an issue for such theories. In the present paper we shall follow a completely different route from that of modified gravity and/or Einstein-æther theories. We shall propose an alternative to these theories in the form of a specific modified matter theory based on an elementary interpretation of the MOND equation (3).

3 Interpretation of the phenomenology of MOND

The physical motivation behind our approach is the striking (and presumably deep) analogy between MOND and the electrostatics of dielectric media\textsuperscript{19}. From electromagnetism in dielectric media we know that the Maxwell-Gauss equation can be written in the two equivalent forms

\[ \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} (\sigma_{\text{free}} + \sigma_{\text{pol}}) \quad \iff \quad \nabla \cdot (\varepsilon \mathbf{E}) = \frac{1}{\varepsilon_0} \sigma_{\text{free}}, \]

where \( \mathbf{E} \) is the electric field, \( \sigma_{\text{free}} \) and \( \sigma_{\text{pol}} \) are the densities of free and polarized (electric) charges respectively, and \( \varepsilon = 1 + \chi_{\varepsilon} \) is the relative permittivity of the dielectric medium. Such an equivalence is only possible because the density of polarized charges reads \( \sigma_{\text{pol}} = -\nabla \cdot \mathbf{P} \), where the polarization field \( \mathbf{P} \) is aligned with the electric field according to \( \mathbf{P} = \varepsilon_0 \chi_{\varepsilon} \mathbf{E} \), the proportionality coefficient \( \chi_{\varepsilon}(\mathbf{E}) \) being known as the electric susceptibility.

By full analogy, one can write the MOND equation (3) in the form of the usual Poisson equation but sourced by some additional distribution of “polarized gravitational masses” \( \rho_{\text{pol}} \) (to be interpreted as dark matter, or a component of dark matter), namely

\[ \nabla \cdot \mathbf{g} = -4\pi G (\rho_{\text{b}} + \rho_{\text{pol}}) \quad \iff \quad \nabla \cdot (\mu \mathbf{g}) = -4\pi G \rho_{\text{b}}. \]

This rewriting stands as long as the mass density of polarized masses appears as the divergence of a vector field, namely takes the dipolar form

\[ \rho_{\text{pol}} = -\nabla \cdot \mathbf{\Pi}, \]

where \( \mathbf{\Pi} \) denotes the (gravitational analogue of the) polarization field. It is aligned with the gravitational field \( \mathbf{g} \) (i.e. the dipolar medium is polarized) according to

\[ \mathbf{\Pi} = -\frac{\chi}{4\pi G} \mathbf{g}. \]

Here the coefficient \( \chi \), which depends on the norm of the gravitational field \( g = |\mathbf{g}| \) in complete analogy with the electrostatics of dielectric media, is related to the MOND function by

\[ \mu = 1 + \chi. \]

Obviously \( \chi \) can be interpreted as a “gravitational susceptibility” coefficient, while \( \mu \) itself can rightly be called a “digravitational” coefficient. It was shown in\textsuperscript{19} that in the gravitational case the sign of \( \chi \) should be negative, in perfect agreement with what MOND predicts; indeed, we have \( \mu < 1 \) in a straightforward interpolation between the MOND and Newtonian regimes, hence \( \chi < 0 \). This finding is in contrast to electromagnetism where \( \chi_{\varepsilon} \) is positive. It can be viewed as
some "anti-screening" of gravitational (baryonic) masses by polarization masses — the opposite
effect of the usual screening of electric (free) charges by polarization charges. Such anti-screening
mechanism results in an enhancement of the gravitational field à la MOND, and offers a very
nice interpretation of the MOND phenomenology. Furthermore, it was pointed out that the
stability of the dipolar dark matter medium requires the existence of some internal force, which
turned out to be simply (in a crude quasi-Newtonian model\textsuperscript{19}) that of an harmonic oscillator.
This force could then be interpreted as the restoring force in the gravitational analogue of a
plasma oscillating at its natural plasma frequency. Finally, it seems from this discussion that the
dielectric interpretation of MOND is deeper than a mere formal analogy. However the model\textsuperscript{19}
is clearly non-viable because it is non-relativistic, and it involves negative gravitational-type
masses and therefore a violation of the equivalence principle at a fundamental level.

4 Relativistic model for the dipolar dark fluid

Here we shall take seriously the physical intuition that MOND has something to do with a
mechanism of gravitational polarization. We shall build a fully relativistic model based on a
matter action in standard general relativity. Note that this means we are changing the point
of view of the original MOND proposal. Instead of requiring a modification of the laws of
gravity in the absence of dark matter, we advocate that the phenomenology of MOND results
from a physically well-motivated mechanism acting on a new type of dark matter, very exotic
compared to standard particle dark matter. Thus, we are proposing a modification of the dark
matter sector rather than a modification of gravity as in TeVeS like theories.

4.1 Action and equations of motion

From the previous discussion, the necessity of endowing dark matter with a new vector field
to build the polarization field is clear. However this vector field will not be expected to be fundamental as in TeVeS like theories. Extending previous work\textsuperscript{20}, our model will be based on a matter action (in Eulerian fluid formalism) in general relativity of the form

\[ S = \int \sqrt{-g} \mathcal{L}[J^\mu, \xi^\mu, \dot{\xi}^\mu, g_{\mu\nu}] \cdot \]  

This action is to be added to the Einstein-Hilbert action for gravity, and to the actions of all the other matter fields. It contains two dynamical variables: (i) a conserved current \(J^\mu = \sigma u^\mu\) satisfying \(\nabla_\mu J^\mu = 0\), where \(u^\mu\) is the normalized four-velocity and \(\sigma = (-J_\nu J^\nu)^{1/2}\) is the rest mass energy density (we pose \(c = 1\) throughout); (ii) the vector field \(\xi^\mu\) representing the dipole four-vector moment carried by the fluid particles. This extra field being dynamical, the Lagrangian will also depend on its covariant derivative \(\nabla_\nu \xi^\mu\); but in our model this dependence will occur only through the covariant time derivative \(\dot{\xi}^\mu\equiv u^\nu \nabla_\nu \xi^\mu\). The Lagrangian explicitly reads (see\textsuperscript{21} for details)

\[ L = \sigma \left[-1 - \Xi + \frac{1}{2} \dot{\xi}_\mu \dot{\xi}^\mu\right] - \mathcal{W}(\Pi_\perp), \]  

where \(\Xi \equiv \{(u_\mu - \dot{\xi}_\mu)(u^\mu - \dot{\xi}^\mu)\}^{1/2}\). The first term is a mass term in the ordinary sense (i.e. the Lagrangian of a pressureless perfect fluid), the second one is inspired by the action of spinning particles in general relativity\textsuperscript{22}, and the third term is a kinetic term for the dipole moment. Finally, the last term represents a potential \(\mathcal{W}\) describing some internal interaction, function of the polarization (scalar) field \(\Pi_\perp = \left(\bot_{\mu\nu} \Pi^\mu \Pi^\nu\right)^{1/2}\), where \(\Pi^\mu = \sigma \xi^\mu\) is the polarization four-vector, and \(\bot_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu\) is the projector orthogonal to the four-velocity.

By varying the action (9)–(10) with respect to both the current \(J^\mu\) and the dipole moment \(\xi^\mu\), we get two dynamical equations: an equation of motion for the dipolar fluid, and an evolution
equation for the dipole moment $\xi^\mu$. From these equations it can be shown that we can impose the constraint $\Xi = 1$ as a particular way of selecting a physically interesting solution, such that the final equations depend only on the space-like projection $\xi_\perp^\mu = \perp_{\mu\nu} \xi^\nu$ of the dipole moment $\xi^\mu$. The final equations we obtain are

$$\dot{u}^\mu = -F^\mu \equiv -\hat{\xi}_\perp^\mu \mathcal{W}', \quad (11)$$

$$\dot{\Omega}^\mu = \frac{1}{\sigma} \nabla^\mu \left( \mathcal{W} - \Pi_\perp \mathcal{W}' \right) - \xi_\perp^\nu R^\mu_{\rho\nu\lambda} u^\rho u^\lambda, \quad (12)$$

where we denote $\Omega^\mu \equiv u^\mu (1 + \xi_\perp \mathcal{W}') + \perp\xi_\perp^\mu$, and employ the notations $\hat{\xi}_\perp^\mu \equiv \xi_\perp^\mu / \xi_\perp$ and $\mathcal{W}' \equiv d\mathcal{W}/d\Pi_\perp$. The motion of the dipolar fluid as given by (11) is non-geodesic, and driven by the internal force $F^\mu$ derived from the potential $\mathcal{W}$. Observe the coupling to the Riemann curvature tensor in the equation of evolution (12) of the dipole moment. By varying the action with respect to the metric $g_{\mu\nu}$ we obtain the stress-energy tensor $T^\mu_{\nu}$. Using the canonical decomposition $T^\mu_{\nu} = r u^\mu u^\nu + \mathcal{P}_\perp^\mu_{\nu} + 2Q^\mu_{\rho\nu} u^\rho + \Sigma^\mu_{\nu}$, we find the energy density $r$, pressure $\mathcal{P}$, heat flux $Q^\mu$ (such that $u^\mu Q^\mu = 0$) and anisotropic stresses $\Sigma^\mu_{\nu}$ ($u^\nu \Sigma^\mu_{\nu} = 0$ and $\Sigma^\nu_{\nu} = 0$) as

$$r = \mathcal{W} - \Pi_\perp \mathcal{W}' + \rho, \quad (13a)$$

$$\mathcal{P} = -\mathcal{W} + \frac{2}{3} \Pi_\perp \mathcal{W}', \quad (13b)$$

$$Q^\mu = \sigma \hat{\xi}_\perp^\mu + \Pi_\perp \mathcal{W}' u^\mu - \Pi_\perp \nabla \xi_\perp^\mu, \quad (13c)$$

$$\Sigma^\mu_{\nu} = \left( \frac{1}{3} \perp^\mu_{\nu} - \hat{\xi}_\perp^\mu \hat{\xi}_\perp^\nu \right) \Pi_\perp \mathcal{W}'. \quad (13d)$$

Here the contribution $\rho$ to the energy density involves a monopolar term $\sigma$ and a dipolar term $-\nabla \lambda \Pi_\perp^1$ which clearly appears as a relativistic generalisation of (6), and will play the crucial role when recovering MOND:

$$\rho = \sigma - \nabla \lambda \Pi_\perp^1. \quad (14)$$

### 4.2 Weak field expansion of the internal potential

The dipolar fluid dynamics in a given background metric, and its influence on spacetime are now known; in the following we shall apply this model to large-scale cosmology and to galactic halos. For both applications we shall need to consider the model in a regime of weak gravity, which will be either first-order perturbations around a Friedman-Lemaître-Robertson-Walker (FLRW) background in cosmology, or the non-relativistic limit for galaxies. A crucial assumption we make is that the potential function $\mathcal{W}$ admits a minimum when the polarization $\Pi_\perp$ is zero, and can be Taylor-expanded around that minimum, with coefficients being entirely specified (modulo an overall factor $G$) by the single surface density scale built from the MOND acceleration $a_0$,

$$\Sigma \equiv \frac{a_0}{2\pi G}. \quad (15)$$

These coefficients in the expansion of $\mathcal{W}$ when $\Pi_\perp \to 0$ will be fine-tuned in order to recover the relevant physics at cosmological and galactic scales. Physically, this expansion corresponds to $\Pi_\perp \ll \Sigma$ and is valid in the weak gravity limit $g \ll a_0$. Clearly, the minimum of $\mathcal{W}$ is nothing but a cosmological constant $\Lambda$, and we find

$$\mathcal{W}(\Pi_\perp) = \frac{\Lambda}{8\pi G} + 2\pi G \Pi_\perp^2 + \frac{8\pi G}{3\Sigma} \Pi_\perp^3 + \mathcal{O}(\Pi_\perp^4). \quad (16)$$

The expansion is thereby determined up to third order inclusively.
Our assumption that the function \( W \) involves the single fundamental scale \( \Sigma \) implies in particular that the cosmological constant \( \Lambda \) should scale with \( G^2 \Sigma^2 \sim a_0^2 \). We thus introduce a dimensionless parameter \( \alpha \) through
\[
\alpha a_0 = \frac{1}{2\pi} \sqrt{\frac{\Lambda}{3}} \equiv a_{\Lambda}. \tag{17}
\]
This parameter represents a conversion factor between \( a_0 \) and the natural acceleration scale \( a_{\Lambda} \) associated with the cosmological constant \( \Lambda \). Posing \( x \equiv \Pi_\perp / \Sigma \), we find that (16) can be recast in the form
\[
W(\Pi_\perp) = 6\pi G \Sigma^2 w(x),
\]
where
\[
w(x) = \alpha^2 \pi^2 + \frac{1}{3} x^2 + \frac{4}{9} x^3 + \mathcal{O}(x^4). \tag{18}
\]
The present model should be considered only as “effective” or phenomenological, in the sense that the weak-gravity expansion (16) or (18) should come from a more fundamental (presumably quantum) underlying theory. Therefore, the coefficients in (18) should not be given by exceedingly large or small numbers (like \( 10^{10} \) or \( 10^{-10} \)), but rather be numerically of the order of one, up to a factor of, say, ten. Hence, we expect that \( \alpha \) itself should be around one, and we see by (17) that the cosmological constant \( \Lambda \) should naturally be numerically of the order of \( a_0^2 \). Notice though that our model does not provide a way to compute the exact value of \( \alpha \). However, one can say that the “cosmic coincidence” (see e.g. 8) between the values of \( a_0 \) and \( a_{\Lambda} \) — with the measured conversion factor being \( \alpha \approx 0.8 \) — finds a natural explanation if dark matter is made of a fluid of polarizable gravitational dipole moments.

5 Recovering the \( \Lambda \)-CDM scenario at cosmological scales

Consider a small perturbation of a background FLRW metric valid between, say, the end of the inflationary era and the recombination. The dipolar fluid is described by its four-velocity \( u^\mu = \bar{u}^\mu + \delta u^\mu \), where \( \delta u^\mu \) is a perturbation of the background comoving four-velocity \( \bar{u}^\mu = (1,0) \), and by its rest mass density \( \sigma = \bar{\sigma} + \delta \sigma \), where \( \delta \sigma \) is a perturbation of the mean cosmological value \( \bar{\sigma} \). The crucial point in our analysis of the dipolar fluid in cosmology is that the background value of the dipole moment field \( \xi_\perp^\mu \) (which is orthogonal to the four-velocity and therefore is space-like) must vanish in order to preserve the spatial isotropy of the FLRW background. We shall thus write \( \xi_\perp^\mu = \delta \xi_\perp^\mu \), and similarly \( \Pi_\perp^\mu = \delta \Pi_\perp^\mu \) for the polarization.

At first perturbation order, the stress-energy tensor with explicit components (13) can be naturally recast, using also (16), in the form \( T^{\mu\nu} = T^{\mu\nu}_{\text{de}} + T^{\mu\nu}_{\text{dm}} \), where the explicit expressions of the dark energy and dipolar dark matter components are
\[
T^{\mu\nu}_{\text{de}} = -\frac{\Lambda}{8\pi G} g^{\mu\nu}, \tag{19}
\]
\[
T^{\mu\nu}_{\text{dm}} = \rho u^\mu u^\nu + 2 Q^{(\mu} u^{\nu)}. \tag{20}
\]
At that order, we find that the dipolar dark matter density reduces to \( \rho \) given by (14). The heat flux in (20) reads as \( Q^\mu = \sigma \xi_\perp^\mu - \Pi_\parallel^\mu \nabla_\lambda u^\lambda \); it is non zero, however it is perturbative because so are both \( \xi_\perp^\mu \) and \( \Pi_\perp^\mu \). The point for our purpose is that at linear perturbation order \( Q^\mu \) can be absorbed into a redefinition of the perturbed four-velocity of the dipolar fluid. Posing \( \tilde{\delta u}^\mu = \delta u^\mu + Q^\mu / \bar{\sigma} \) and introducing the effective four-velocity \( \tilde{u}^\mu = \bar{u}^\mu + \tilde{\delta u}^\mu \), we find that at first order the dipolar dark matter fluid is described by the stress-energy tensor
\[
T^{\mu\nu}_{\text{dm}} = \rho \tilde{u}^\mu \tilde{u}^\nu, \tag{21}
\]
which is that of a perfect fluid with four-velocity \( \tilde{u}^\mu \), vanishing pressure and energy density (14). In the linear cosmological regime, the dipolar fluid therefore behaves as standard cold dark
matter (a pressureless fluid) plus standard dark energy (a cosmological constant). Adjusting
the background value $\sigma$ so that $\Omega_{dm} \simeq 23\%$, the model is thus consistent with the standard $\Lambda$-CDM
scenario and the cosmological observations of the CMB fluctuations (see\textsuperscript{21} for more details).

6 Recovering the MOND phenomenology at galactic scales

Next, we turn to the study of the dipolar dark fluid in a typical galaxy at low redshift. We
have to consider the non-relativistic limit ($c \to +\infty$) of the model; we consistently neglect all
relativistic terms $O(c^{-2})$. It is straightforward to check that in the non-relativistic limit the
equation of motion (11) of the dipolar fluid reduces to
\[
\frac{dv}{dt} = g - \hat{\Pi}_\perp W',
\]
where $g$ is the local gravitational field generated by both the baryonic matter in the galaxy and
the dipolar dark matter. Applying next the standard minimal coupling to gravity in general
relativity, we find that the gravitational field obeys the Poisson equation
\[
\nabla \cdot g = -4\pi G (\rho_b + \rho),
\]
where $\rho_b$ and $\rho$ are respectively the baryonic and dipolar dark matter mass densities. From (14)
the dipolar dark matter density reduces in the non-relativistic limit to
\[
\rho = \sigma - \nabla \cdot \Pi_\perp.
\]
The first term is a usual monopolar contribution: the rest mass density $\sigma$ of the fluid, while
the second term is the dipolar contribution, and can be interpreted as coming from the fluid's
internal energy. Here $\Pi_\perp$ denotes the spatial components of the polarization.

In order to recover MOND, we need two things: (i) to find a mechanism for the alignment
of the polarization field $\Pi_\perp$ with the gravitational field $g$, so that a relation similar to (7)
will apply; (ii) to justify that the rest mass density $\sigma$ of dipole moments in (24) is small with
respect to the baryonic density $\rho_b$, hence the galaxy will mostly appear as baryonic in MOND
fits of rotation curves. We have proposed in\textsuperscript{21} a single mechanism able to answer positively
these two points. We call it the hypothesis of weak clustering of dipolar dark matter; it is
an hypothesis because it has been conjectured but not proved, and should be checked using
numerical simulations. The weak clustering hypothesis is motivated by a solution of the full set
of equations describing the non-relativistic motion of dipolar dark matter in a typical baryonic
galaxy whose mass distribution $\rho_b$ is spherically symmetric (see the Appendix in\textsuperscript{21}). This
particular solution corresponds to an equilibrium configuration in spherical symmetry, for which
$\mathbf{v} = 0$ and $\sigma = \sigma_0(r)$. The dipole moments remain at rest because the gravitational field $g$
balanced by the internal force $\mathbf{F} = \Pi_\perp W'$; for that solution the right-hand-side of (22) vanishes.

During the cosmological evolution we expect that the dipolar medium will not cluster much
because the internal force may balance part of the local gravitational field generated by an
overdensity. The dipolar dark matter density contrast in a typical galaxy at low redshift should
thus be small, at least smaller than in the standard $\Lambda$-CDM scenario. Hence $\sigma \ll \rho_b$, and we
could even envisage that $\sigma$ stays around its mean cosmological value, $\sigma \sim \bar{\sigma} \ll \rho_b$. Now, because
of its size and typical time-scale of evolution, a galaxy is almost unaffected by the cosmological
expansion of the Universe. The cosmological mass density $\bar{\sigma}$ of the dipolar dark matter is not
only homogeneous, but also almost constant in this galaxy. The continuity equation reduces to
$\nabla \cdot (\bar{\sigma} \mathbf{v}) \simeq 0$, and the most simple solution corresponds to a static fluid verifying $\mathbf{v} \simeq 0$. By (22)
we see that the polarization field $\Pi_\perp$ is then aligned with the gravitational field $g$, namely
\[
g \simeq \Pi_\perp W'.
\]
On the other hand, because $\sigma \ll \rho_b$ by the same mechanism, we observe that the gravitational field equation (23) with (24) becomes
\[ \nabla \cdot (g - 4\pi G \Pi_\perp) \simeq -4\pi G \rho_b, \]
which according to (5) is found to be rigorously equivalent to the MOND equation. Finally, by inserting the expression of the potential (16) into (25) and comparing with the defining equation (7), we readily find the MOND behaviour of the gravitational susceptibility coefficient as
\[ \chi(g) = -1 + \frac{g}{a_0} + O(g^2), \]
in complete agreement with (2). We can thus state that the dipolar fluid described by the action (9)–(10) explains the phenomenology of MOND in a typical galaxy. Note also that in this model there is no problem with the light deflection by galaxy clusters. Indeed the standard general relativistic coupling to gravity implies the usual formula for the bending of light.

To conclude, the present model reconciles in some sense the observations of dark matter on cosmological scales, where the evidence is for cold dark matter, and on galactic scales, which is the realm of MOND. In addition, it offers a nice unification between the dark energy in the form of $\Lambda$ and the dark matter in galactic halos. More work should be done to test the model, either by studying second-order perturbations in cosmology, or by computing numerically the non-linear growth of perturbations and comparing with large-scale structures, or by studying the intermediate scale of clusters of galaxies.

References