Effective field theory for perturbations in dark energy & modified gravity

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arXiv: 1203.0398
The problem...

- General Relativity + FRW + standard model particles + observational data: *inconsistent*
  - ... invent dark energy ~70%.
- Or perhaps GR is not the right gravitational theory for cosmological scales...
  - c.f. Newton & Mercury/Sun system
- **Models of dark energy:** $\Lambda$, quintessence, k-essence, elastic dark energy, ...
- **Modified gravity:** $F(R)$, Horndeski, galileon, Gauss-Bonnet, Aether, TeVeS, ...
  - MG... obtain different gravitational potential for the same matter content
- Lagrangian engineering
Generalized gravitational field equations

* All modified gravity & dark energy theories have gravitational field equations which can be written as

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + U_{\mu\nu}$$

* Stems from an action:

$$\mathcal{L}_{\text{grav}} = \mathcal{L}_{\text{GR}} + \mathcal{L}_{\text{known}} + \mathcal{L}_{\text{dark}}$$

* At perturbed order: structures...

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} + \delta U_{\mu\nu}$$

* Q: How do we write down the allowed, consistent modifications to the gravity field equations? A: Lagrangian for the dark sector perturbations
**Lagrangian for perturbations**

Pick field content \[ \mathcal{L} = \mathcal{L}(X^{(A)}) \]

The Lagrangian for perturbations is a **quadratic functional** in perturbations of these fields...

\[ \mathcal{L}_{\{2\}} = \mathcal{L}_{\{2\}}[\delta X^{(A)}] \]

**Quadratic functional** is written as...

\[ \mathcal{L}_{\{2\}} = \sum_{A=1}^{N} \sum_{B=1}^{N} G_{AB}^{\{0\}} \delta X^{(A)} \delta X^{(B)} \]

\[ G_{AB}^{\{0\}} = G_{AB}^{\{0\}}(X^{(C)}) \quad \text{N(N+1)/2 functions of background field variables} \]

Now use to calculate equations of motion, energy-momentum tensor, ...
Lagrangian for perturbations

The Lagrangian for perturbations is equivalent to the *second measure-weighted variation* of the action...

\[
\delta^2 S = \int d^4x \sqrt{-g} \left[ \frac{1}{\sqrt{-g}} \delta^2 (\sqrt{-g} \mathcal{L}) \right] = \int d^4x \sqrt{-g} \mathcal{L}_{\{2\}}
\]

This allows us to

\[\diamondsuit^2 \mathcal{L}\]

(1) “dream up” a Lagrangian for perturbations,

(2) explicitly calculate for some known theory... compare with established theories

Map from general parameterization to established theories, but we are not limited by them!
**Gravitational effects**

Source to perturbed gravitational field equations:

\[ \delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} + \delta U_{\mu\nu} \]

Perturbed dark energy momentum tensor is a linear functional...

\[ \delta U^{\mu\nu} = \delta U^{\mu\nu}[\delta X^{(A)}] \quad \delta U^{\mu\nu} = \sum_{A=1}^{N} \hat{W}^{\mu\nu}_A \delta X^{(A)} \]

Calculate via functional derivative w.r.t perturbed metric...

\[ \delta U^{\mu\nu} = -\frac{1}{2} \left[ 4 \frac{\delta}{\delta h_{\mu\nu}} \mathcal{L}_{\{2\}} + U^{\mu\nu} g^{\alpha\beta} h_{\mu\nu} \right] \]

Satisfies conservation equation:

\[ \delta (\nabla_\mu U^{\mu\nu}) = 0 \]

\[ \nabla_\mu \delta U^{\mu\nu} + \frac{1}{2} \left[ U^{\mu\nu} g^{\alpha\beta} - U^{\alpha\beta} g^{\mu\nu} + 2g^{\nu\beta} U^{\alpha\mu} \right] \nabla_\mu h_{\alpha\beta} = 0. \]
Nothing extra

Field content: just the metric

**Lagrangian for perturbations**

\[ \mathcal{L}_{\{2\}} = \frac{1}{8} \mathcal{W}^{\mu\nu\alpha\beta} \delta g_{\mu\nu} \delta g_{\alpha\beta} \]

Gravitational effects...

\[ \delta U^{\mu\nu} = -\frac{1}{2} \left[ \mathcal{W}^{\mu\nu\alpha\beta} + U^{\mu\nu} g^{\alpha\beta} \right] \delta g_{\alpha\beta} \]

Elastic dark energy, Feirz-Pauli & massive gravity in GR,...
Scalar fields

\[ \mathcal{L} = \mathcal{L}(g_{\mu\nu}, \phi, \nabla_{\mu}\phi) \]

\[ \mathcal{L}_{\{2\}} = \mathcal{L}_{\{2\}}(\delta g_{\mu\nu}, \delta \phi, \nabla_{\mu}\delta \phi) \]

Lagrangian for perturbations

\[
\mathcal{L}_{\{2\}} = A\delta\phi\delta\phi + B^\mu\delta\phi\nabla_\mu\delta\phi + \frac{1}{2}C^{\mu\nu}\nabla_\mu\delta\phi\nabla_\nu\delta\phi \\
+ \frac{1}{4} \left[ \mathcal{V}^{\mu\nu}\delta\phi\delta g_{\mu\nu} + \mathcal{Y}^{\alpha\mu\nu}\nabla_\alpha\delta\phi\delta g_{\mu\nu} + \frac{1}{2}\mathcal{W}^{\mu\nu\alpha\beta}\delta g_{\alpha\beta}\delta g_{\mu\nu} \right]
\]

Gravitational effects...

\[
\delta U^{\mu\nu} = -\frac{1}{2} \left[ \mathcal{V}^{\mu\nu}\delta\phi + \mathcal{Y}^{\alpha\mu\nu}\nabla_\alpha\delta\phi + \mathcal{W}^{\mu\nu\alpha\beta}\delta g_{\alpha\beta} + U^{\mu\nu} g^{\alpha\beta}\delta g_{\alpha\beta} \right]
\]

Quintessence, k-essence, Lorentz violating theories,...
**Imposing symmetry on the background**

e.g. isotropy of spatial sections...

everything becomes compatible with FRW

\[(3+1)\text{ decomposition} \quad g_{\mu\nu} = \gamma_{\mu\nu} - u_\mu u_\nu \quad \gamma_{\mu\nu} u^\nu = 0 \quad u_\mu u^\mu = -1\]

Coefficients in Lagrangian become:

\[
\mathcal{W}_{\mu\nu\alpha\beta} = A_W u_\mu u_\nu u_\alpha u_\beta + B_W \left( \gamma_{\mu\nu} u_\alpha u_\beta + \gamma_{\alpha\beta} u_\mu u_\nu \right) + 2C_W \left( \gamma_{\mu(\alpha} u_{\beta)} u_\nu + \gamma_{\nu(\alpha} u_{\beta)} u_\mu \right) + \mathcal{E}_{\mu\nu\alpha\beta},
\]

\[
\mathcal{E}_{\mu\nu\alpha\beta} = D_W \gamma_{\mu\nu} \gamma_{\alpha\beta} + 2E_W \gamma_{\mu(\alpha} \gamma_{\beta)\nu}.\]
Imposing symmetry on the background

e.g. isotropy of spatial sections...
everything becomes compatible with FRW

\[ (3+1) \text{ decomposition} \quad g_{\mu \nu} = \gamma_{\mu \nu} - u_\mu u_\nu \quad \gamma_{\mu \nu} u^\nu = 0 \quad u_\mu u^\mu = -1 \]

Coefficients in Lagrangian become:

\[ \mathcal{W}_{\mu \nu \alpha \beta} = A_{\mathcal{W}} u_\mu u_\nu u_\alpha u_\beta + B_{\mathcal{W}} \left( \gamma_{\mu \nu} u_\alpha u_\beta + \gamma_{\alpha \beta} u_\mu u_\nu \right) \]
\[ + 2C_{\mathcal{W}} \left( \gamma_{\mu (\alpha } u_{\beta ) \nu} + \gamma_{\nu (\alpha } u_{\beta ) \mu} \right) + \mathcal{E}_{\mu \nu \alpha \beta}, \]

\[ \mathcal{E}_{\mu \nu \alpha \beta} = D_{\mathcal{W}} \gamma_{\mu \nu} \gamma_{\alpha \beta} + 2E_{\mathcal{W}} \gamma_{\mu (\alpha} \gamma_{\beta ) \nu}. \]
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Coefficients in Lagrangian become:

\[\mathcal{W}_{\mu \nu \alpha \beta} = A_\mathcal{W} u_\mu u_\nu u_\alpha u_\beta + B_\mathcal{W} \left( \gamma_{\mu \nu} u_\alpha u_\beta + \gamma_{\alpha \beta} u_\mu u_\nu \right) + 2 C_\mathcal{W} \left( \gamma_{\mu (\alpha} u_{\beta)} u_\nu + \gamma_{\nu (\alpha} u_{\beta)} u_\mu \right) + \mathcal{E}_{\mu \nu \alpha \beta},\]

\[\mathcal{E}_{\mu \nu \alpha \beta} = D_\mathcal{W} \gamma_{\mu \nu} \gamma_{\alpha \beta} + 2 E_\mathcal{W} \gamma_{\mu (\alpha} \gamma_{\beta) \nu},\]

\[\mathcal{A} = A_A,\]

\[\mathcal{B}^\mu = A_B u^\mu,\]

\[\mathcal{C}_{\mu \nu} = A_C u_\mu u_\nu + B_C \gamma_{\mu \nu},\]

\[\mathcal{Y}_{\mu \nu} = A_Y u_\mu u_\nu + B_Y \gamma_{\mu \nu},\]

\[\mathcal{Y}_{\alpha \mu \nu} = A_Y u_\alpha u_\mu u_\nu + B_Y u_\alpha \gamma_{\mu \nu} + 2 C_Y \gamma_{\alpha (\mu u_\nu)}.\]
Imposing symmetry on the background

e.g. isotropy of spatial sections...
everything becomes compatible with FRW

(3+1) decomposition \[ g_{\mu\nu} = \gamma_{\mu\nu} - u_\mu u_\nu \quad \gamma_{\mu\nu} u^\nu = 0 \quad u_\mu u^\mu = -1 \]

Coefficients in Lagrangian become:

\[ \mathcal{W}_{\mu\nu\alpha\beta} = A_W u_\mu u_\nu u_\alpha u_\beta + B_W \left( \gamma_{\mu\nu} u_\alpha u_\beta + \gamma_{\alpha\beta} u_\mu u_\nu \right) + 2C_W \left( \gamma_{(\alpha} u_{\beta)} u_\nu + \gamma_{(\alpha} u_{\beta)} u_\mu \right) + \mathcal{E}_{\mu\nu\alpha\beta}, \]

\[ \mathcal{E}_{\mu\nu\alpha\beta} = D_W \gamma_{\mu\nu} \gamma_{\alpha\beta} + 2E_W \gamma_{(\alpha} \gamma_{\beta)\nu}. \]

\[ \mathcal{A} = A_A, \]

\[ \mathcal{B}^\mu = A_B u^\mu, \]

\[ \mathcal{C}_{\mu\nu} = A_C u_\mu u_\nu + B_C \gamma_{\mu\nu}, \]

\[ \mathcal{V}_{\mu\nu} = A_Y u_\mu u_\nu + B_Y \gamma_{\mu\nu}, \]

\[ \mathcal{V}_{\alpha\mu\nu} = A_Y u_\alpha u_\mu u_\nu + B_Y u_\alpha \gamma_{\mu\nu} + 2C_Y \gamma_{\alpha(\mu u_\nu)}. \]
Imposing symmetry on the background

e.g. isotropy of spatial sections...

everything becomes compatible with FRW

(3+1) decomposition \( g_{\mu\nu} = \gamma_{\mu\nu} - u_{\mu} u_{\nu} \) \( \gamma_{\mu\nu} u^{\nu} = 0 \) \( u_{\mu} u^{\mu} = -1 \)

Coefficients in Lagrangian become:

\[ \mathcal{N}_{\mu\nu\alpha\beta} = A_{\mathcal{N}} u_{\mu} u_{\nu} u_{\alpha} u_{\beta} + B_{\mathcal{N}} \left( \gamma_{\mu\nu} u_{\alpha} u_{\beta} + \gamma_{\alpha\beta} u_{\mu} u_{\nu} \right) \]

\[ + 2C_{\mathcal{N}} \left( \gamma_{\mu(\alpha} u_{\beta)} u_{\nu} + \gamma_{\nu(\alpha} u_{\beta)} u_{\mu} \right) + \mathcal{E}_{\mu\nu\alpha\beta}, \]

\[ \mathcal{E}_{\mu\nu\alpha\beta} = D_{\mathcal{E}} \gamma_{\mu\nu} \gamma_{\alpha\beta} + 2E_{\mathcal{E}} \gamma_{\mu(\alpha} \gamma_{\beta)\nu}. \]

Impose some “theory” structure:

\[ \mathcal{L} = \mathcal{L}(\phi, \chi) \] reduce 14 -> 3

\[ \mathcal{V}_{\mu\nu} = A_{\mathcal{V}} u_{\mu} u_{\nu} + B_{\mathcal{V}} \gamma_{\mu\nu}, \]

\[ \mathcal{Y}_{\alpha\mu\nu} = A_{\mathcal{Y}} u_{\alpha} u_{\mu} u_{\nu} + B_{\mathcal{Y}} u_{\alpha} \gamma_{\mu\nu} + 2C_{\mathcal{Y}} \gamma_{\alpha(\mu} u_{\nu)} \].
This means that we are setting the components of the Eulerian perturbed metric to

\[ \xi^\mu_{\text{pert}} = 0. \]

In this section we provide explicit expressions for the components of the perturbed conservation equation specialized to the case of an FRW background. We will pay special attention to the scalar field theory where we will show how the vector field will evolve with respect to time; for conformal time coefficients one should replace

\[ \frac{d}{d\tau} \rightarrow \frac{\dot{\phi}}{H} \]

with \( H \) the Hubble parameter. It is interesting to realize that the theories have presented are: way elastic dark energy, wby generic kinetic scalar field theory, wcy hyperbolic scalar field, wdy canonical scalar field theory. It is interesting to realize that the theories we have presented are: way elastic dark energy, wby generic kinetic scalar field theory, wcy hyperbolic scalar field, wdy canonical scalar field theory.

### Table 1

<table>
<thead>
<tr>
<th>Function</th>
<th>(a) EDE</th>
<th>(b) ( \mathcal{L} = \mathcal{L}(\phi, \mathcal{X}) )</th>
<th>(c) ( \mathcal{L} = F(\mathcal{X}) )</th>
<th>(d) ( \mathcal{L} = \mathcal{X} - V(\phi) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_V )</td>
<td>0</td>
<td>(-2(\mathcal{L}<em>x \dot{\phi}^2 - \mathcal{L}</em>\phi))</td>
<td>0</td>
<td>(-2V'')</td>
</tr>
<tr>
<td>( B_V )</td>
<td>0</td>
<td>(-2\mathcal{L}_\phi)</td>
<td>0</td>
<td>(2V')</td>
</tr>
<tr>
<td>( A_Y )</td>
<td>0</td>
<td>(-2(\mathcal{L}<em>x \dot{\phi}^2 + \mathcal{L}</em>\phi))</td>
<td>(-2(F''\dot{\phi}^2 + F'\dot{\phi}))</td>
<td>(-2\dot{\phi})</td>
</tr>
<tr>
<td>( B_Y )</td>
<td>0</td>
<td>(-2\mathcal{L}_x \dot{\phi})</td>
<td>(-2F'\dot{\phi})</td>
<td>(-2\dot{\phi})</td>
</tr>
<tr>
<td>( C_Y )</td>
<td>0</td>
<td>(2\mathcal{L}_x \dot{\phi})</td>
<td>(2F'\dot{\phi})</td>
<td>(2\dot{\phi})</td>
</tr>
<tr>
<td>( A_W )</td>
<td>(-\rho)</td>
<td>((-\mathcal{L}_x \phi \dot{\phi}^4 + 2\rho + P))</td>
<td>(-(F''\dot{\phi}^4 + 2\rho + P))</td>
<td>(-(2\rho + P))</td>
</tr>
<tr>
<td>( B_W )</td>
<td>(P)</td>
<td>(-\rho)</td>
<td>(-\rho)</td>
<td>(-\rho)</td>
</tr>
<tr>
<td>( C_W )</td>
<td>(-P)</td>
<td>(\rho)</td>
<td>(\rho)</td>
<td>(\rho)</td>
</tr>
<tr>
<td>( D_W )</td>
<td>(\beta - P - \frac{2}{3}\mu)</td>
<td>(-P)</td>
<td>(-P)</td>
<td>(-P)</td>
</tr>
<tr>
<td>( E_W )</td>
<td>(\mu + P)</td>
<td>(P)</td>
<td>(P)</td>
<td>(P)</td>
</tr>
</tbody>
</table>

In the table above, \( \mathcal{L} \) denotes the Lagrangian density, \( \mathcal{X} \) a function of the metric, \( \phi \) a scalar field, and \( V(\phi) \) the potential. The terms \( \mathcal{L}_x \) and \( \mathcal{L}_\phi \) denote partial derivatives with respect to the metric and the scalar field, respectively. The dot \( \dot{\phi} \) denotes differentiation with respect to cosmic time, and \( H \) is the Hubble parameter. The table shows how the Lagrangian density \( \mathcal{L} \) and the potential \( V(\phi) \) contribute to the conservation equation in the presence of a scalar field, with \( \mathcal{L}_x \) and \( \mathcal{L}_\phi \) representing the part of the Lagrangian that is linear in \( \dot{\phi} \) and \( \phi \), respectively.
Parameterizing entropy \( \delta P = \omega \delta \rho + P \Gamma \)

\[
\omega \Gamma = (\alpha - \omega) \left[ \delta - 3\mathcal{H} \beta (1 + \omega) \theta \right]
\]

Standard: Weller & Lewis: \( \alpha = c_s^2 \quad \beta = 1 \)

**Physically, what does \( \alpha 
eq 1 \) mean?**

Is \( \alpha \) always a sound speed?

\( \alpha \) is neither group nor phase velocity of waves

\[
\alpha = \left(1 + 2\chi \frac{\mathcal{L},xx}{\mathcal{L},x} \right)^{-1},
\]

\[
\beta = \frac{2aL,\phi}{3\mathcal{H}L,x\sqrt{2\chi}} \left[1 + \chi \left(\frac{\mathcal{L},xx}{\mathcal{L},x} - \frac{\mathcal{L},x\phi}{\mathcal{L},\phi}\right)\right] \frac{\alpha}{\alpha - \omega}.
\]

**Effective metric** that scalar field perturbations “feel”...

\[
C^{\mu\nu} = \mathcal{L},x g^{\mu\nu} + \mathcal{L},x \alpha^{-1} (\alpha - 1) u^\mu u^\nu
\]
Scope of using the Lagrangian for perturbations

**Formalism**

- The Lagrangian for perturbations
- The perturbed dark energy-momentum tensor
- Lagrangian & Eulerian perturbations (Stuckleberg completion/deformation vector)

**Applications**

- Cosmological perturbations
- Entropy & anisotropic stresses
- Massive gravity
- Modified gravity

**Examples**

- Nothing extra
- Scalar & vector fields (aether, TeVeS, ...)
- “high order” derivative theories (galileon, F(R), Gauss-Bonnet, ...)
Summary

• Construct coherent consistent modifications to the gravitational field equations at perturbed order

• All freedom inside “background” tensors

• Encompass theories never before considered: \( \mathcal{L}_{[2]} \) needs scalars in background \textit{and} perturbed field variables: more freedom!

• Encompass massive gravity & “high derivative” theories: e.g. galileon, Horndeski, Brans-Dicke.

• In a model independent way compute cosmological observables (CMB, lensing, P(k), ...): rule in/out before requiring the actual theory!

• Impose symmetry on background (e.g. isotropy... compatible with FRW)... allows split of BG tensors
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