Post-Newtonian higher-order spin effects in inspiraling compact binaries

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Work in collaboration with:
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Part I

Astrophysical sources of GW

Coalescence of compact objects binaries (black holes/neutron stars)

Spin effects:
- Aligned: affect the orbital phasing
- Misaligned: precession
Part I

Motivation for accurate PN templates

Advanced LIGO-VIRGO band:
- NS-NS binary: ~10000 cycles
- BH-NS binary: ~3000 cycles
- BH-BH binary: ~600 cycles

Templates ingredients:
- Phasing for circular orbits: $E(\omega), \mathcal{F}(\omega)$
- GW polarizations (modes): $h_+, h_\times (h_{lm})$
- Precessional dynamics in presence of spins: $\dot{S}_1, \dot{S}_2, \ell$

Matched filtering:
$$(x|y) = 4\text{Re} \int_0^\infty df \frac{\tilde{x}(f)\tilde{y}^*(f)}{\tilde{S}_n(f)}$$

PN inspiral

High order PN contributions needed for accurate data analysis

PN approximant

GW Phase

Balance equation
$$\mathcal{F} = -dE/dt$$

NR merging-ringdown
### Part I

**Post-Newtonian results: where do we stand?**

#### Dynamics

<table>
<thead>
<tr>
<th>Leading</th>
<th>Known</th>
</tr>
</thead>
<tbody>
<tr>
<td>NS</td>
<td>N 4PN (ADM)</td>
</tr>
<tr>
<td>SO</td>
<td>1.5PN 3.5PN (ADM, H)</td>
</tr>
<tr>
<td>SS</td>
<td>2PN 3PN (SS) - 4PN (S1S2) (ADM, EFT, H)</td>
</tr>
<tr>
<td>SSS</td>
<td>3.5PN 3.5PN (ADM/EFT, H)</td>
</tr>
<tr>
<td>SSSS</td>
<td>4PN 4PN (ADM/EFT)</td>
</tr>
</tbody>
</table>

ADM: reduced Hamiltonian in ADM approach  
EFT: effective field theory  
H: harmonic coordinates MPN method  

#### Energy flux

<table>
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</thead>
<tbody>
<tr>
<td>NS</td>
<td>N 3.5PN (H)</td>
</tr>
<tr>
<td>SO</td>
<td>1.5PN 3.5PN+4PN (H)</td>
</tr>
<tr>
<td>SS</td>
<td>2PN 3PN (SS, S1S2) (partial EFT, H)</td>
</tr>
<tr>
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<td>3.5PN 3.5PN (H)</td>
</tr>
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</tbody>
</table>

#### Full waveform

<table>
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<th>Known</th>
</tr>
</thead>
<tbody>
<tr>
<td>NS</td>
<td>N 3PN (3.5PN) (H)</td>
</tr>
<tr>
<td>S</td>
<td>1PN 2PN (H)</td>
</tr>
</tbody>
</table>

1PN \( \sim \frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \)
Part II

Harmonic coordinates and basics of the method

Einstein equations in harmonic coordinates

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]

Harmonic gauge

\[ h^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} - \eta^{\mu\nu} \]
\[ \partial_{\nu} h^{\mu\nu} = 0 \]

Einstein equations - can be iterated

\[ \Box h^{\mu\nu} = \frac{16\pi G}{c^4} (-g) T^{\mu\nu} + \Lambda^{\mu\nu} \]

\( \Lambda^{\mu\nu}(h^2, h^3, \ldots) \) encodes all non-linearities in \( h \)

Near-zone PN iteration and wave generation

- PN retardation expansion of \( \Box^{-1}_R \) in the near-zone \( r \ll \lambda \)
- Iterative multipolar solution in vacuum \( r > r_{\text{source}} \)

Modeling spins for compact objects

Compact objects as point particles (Dirac deltas)

Model of point particles with spin

UV regularization scheme

- Hadamard regularization
- Dimensional regularization

Higher orders in spin

Asymptotic matching

PN near-zone iteration of field equations

MPM wave generation formalism in vacuum

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Part II  
Point particles with spin: Lagrangian formalism

Geometric definitions

\[ z^{ho}(\tau) \]

\[ u^\mu = \frac{dz^\mu}{d\tau} \quad \text{: 4-velocity} \]

\[ \epsilon_A^\mu \] : field tetrad

\[ \epsilon_A^\mu \] : tetrad attached to the body

\[ \Lambda_A^a \] : Lorentz matrices, 6 internal degrees of freedom

\[ \Omega^{\mu\nu} \equiv \epsilon^A_{~\alpha} \frac{D\epsilon_A^{\alpha}}{d\tau} \quad \text{: rotation coefficients (antisymmetric)} \]

Ansatz for the Lagrangian

\[ S = \int d\tau \left[ u^\mu, \Omega^{\mu\nu}, g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\lambda R_{\mu\nu\rho\sigma}, \ldots \right] \]

Finite size effects
Part II
Lagrangian formalism: conjugate moments

Ansatz for the Lagrangian

\[ S = \int \mathrm{d}\tau \ L \left[ u^\mu, \Omega^{\mu\nu}, g_{\mu\nu}, R_{\mu\nu\rho\sigma}, \nabla_\lambda R_{\mu\nu\rho\sigma}, \ldots \right] \]

Conjugate Moments

Linear momentum: \( p_\mu \equiv \frac{\partial L}{\partial u^\mu} \)

Spin tensor: \( S_{\mu\nu} \equiv 2 \frac{\partial L}{\partial \Omega^{\mu\nu}} \)

Spin supplementary condition: \( p_\mu S^{\mu\nu} = 0 \) (impose 3 conditions)

Multipolar Moments

Quadrupolar moment: \( J_{\mu\nu\rho\sigma} \equiv -6 \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}} \) \( (S^2 \text{ if spin-induced}) \)

Octupolar moment: \( J_{\lambda\mu\nu\rho\sigma} \equiv -12 \frac{\partial L}{\partial \nabla_\lambda R_{\mu\nu\rho\sigma}} \) \( (S^3 \text{ if spin-induced}) \)

... and higher orders

Finite size effects

Spin supplementary condition:

\( p_\mu S^{\mu\nu} = 0 \) (impose 3 conditions)

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... and higher orders
Part II

Lagrangian formalism: equations of motion/precession

Equation of motion

covariant variation of the worldline: \( \delta z^\rho \partial_\rho L = \delta z^\rho \nabla_\rho L \)

\[
\frac{dp_\mu}{d\tau} = -\frac{1}{2} R_{\mu\nu\rho\sigma} u^\nu S^\rho^\sigma - \frac{1}{6} J^{\lambda\nu\rho\sigma} \nabla_\mu R_{\lambda\nu\rho\sigma} - \frac{1}{12} J^{\tau\lambda\nu\rho\sigma} \nabla_\mu \nabla_\tau R_{\lambda\nu\rho\sigma}
\]

Immediate generalization to higher orders

Equation of precession

variation of rotational degrees of freedom: \( \delta \theta^{ab} \equiv \Lambda^A_a \delta \Lambda_A^b \)

\[
\frac{dS^{\mu\nu}}{d\tau} = \Omega^\mu_\rho S^{\nu\rho} - \Omega^\nu_\rho S^{\mu\rho}
\]

- Valid at any multipolar order
- Conserved spin norm, independently of the SSC: \( s^2 \equiv S_{\mu\nu} S^{\mu\nu}/2 = \text{const} \)

Imposing that the Lagrangian is a scalar:

\[
\frac{dS^{\mu\nu}}{d\tau} = 2p^{[\mu} u^{\nu]} + \frac{4}{3} R^{[\mu}_{\lambda\rho\sigma} J^{\nu]} \lambda\rho\sigma + \frac{2}{3} \nabla^\lambda R^{[\mu}_{\tau\rho\sigma} J^{\nu]}_{\lambda} \tau\rho\sigma + \frac{1}{6} \nabla^{[\mu} R_{\lambda\tau\rho\sigma} J^{\nu]} \lambda\tau\rho\sigma
\]
Part II

Lagrangian formalism: stress-energy tensor

Variation $\delta g_{\mu\nu}$

Defining the world line density: $w = \int d\tau \delta^4(x - z)/\sqrt{-g}$

**Pole-dipole terms:**

$$T_{\text{pole-dipole}}^{\mu\nu} = p^{(\mu} u^{\nu)} w - \nabla_\rho \left[ S^{\rho(\mu} u^{\nu)} w \right]$$

**Quadrupole terms:**

$$T_{\text{quad}}^{\mu\nu} = \frac{1}{3} R^{(\mu}_{\lambda\rho\sigma} J^{\nu)} \lambda\rho\sigma w - \nabla_\rho \nabla_\sigma \left[ \frac{2}{3} J^{\rho(\mu\nu)} \sigma w \right]$$

**Octupole terms:**

$$T_{\text{oct}}^{\mu\nu} = \left[ \frac{1}{6} \nabla^\lambda \left( -\frac{1}{6} R^{(\mu}_{\xi\lambda\rho} J^{\nu)} \lambda\rho\sigma \xi\rho\sigma + \frac{1}{12} R^{(\mu}_{\xi\tau\rho\sigma} J^{\nu)} \xi\tau\rho\sigma \right) \right] w$$

$$+ \nabla_\rho \left\{ \left[ -\frac{1}{6} R^{(\mu}_{\xi\lambda\sigma} J^{\rho|\nu)} \xi\lambda\sigma - \frac{1}{3} R^{(\mu}_{\xi\lambda\sigma} J^{\nu)} \rho\xi\lambda\sigma + \frac{1}{3} R^{\rho}_{\xi\lambda\sigma} J^{(\mu\nu)} \xi\lambda\sigma \right] w \right\}$$

$$+ \nabla_\lambda \nabla_\rho \nabla_\sigma \left[ \frac{1}{3} J^{\sigma\rho(\mu\nu)} \lambda w \right]$$

(No direct generalization)
Part II
Lagrangian formalism: spin-induced moments

Representing spin-induced structure

- Elimination of $R_{\mu\nu}$ in the Lagrangian: $C_{\mu\nu\rho\sigma}$
  write all possible couplings with the Weyl tensor

- Write the couplings directly with the spin tensor and using the SSC $p_\mu S^{\mu\nu} = 0$

Spin-induced moments

- Quadrupole:
  $J^{\mu\nu\rho\sigma} = -\frac{3\kappa}{m} u^{[\mu} \Theta^{\nu]} [\rho \sigma]$  \[ \Theta^{\mu\nu} \equiv S^{\mu\lambda} S^{\nu\lambda} \]

- Octupole:
  $J^{\lambda\mu\nu\rho\sigma} = \frac{\lambda}{4m^2} \left[ \Theta^{\lambda [\mu} u^{\nu]} S^{\rho\sigma} + \Theta^{\lambda [\rho} u^{\sigma]} S^{\mu\nu} \\
  - \Theta^{[\lambda} S^{\mu\nu]} [\rho u^\sigma] - \Theta^{[\lambda} S^{\sigma]} [\mu u^\nu] \\
  - S^{\lambda [\mu} \Theta^{\nu]} [\rho u^\sigma] - S^{\lambda [\rho} \Theta^{\sigma]} [\mu u^\nu] \right]$  

Polarizability constants

- $\kappa, \lambda$
  to be determined

- By matching to a Kerr black hole
- Numerically for neutron stars

Generalization at all orders in spin (leading order in the Weyl tensor)

[Levi&Steinhoff 15]
Part III

Results and checks

Summary

- Symbolic computation: Mathematica®, xAct [Martin-Garcia], PNComBin [Faye]
- 3.5PN spin-orbit dynamics and flux/phasing (NNLO)
- 4PN spin-orbit tail terms in the flux/phasing (NLO for the tails)
- 3PN spin-spin dynamics and flux/phasing (NLO)
- 3.5PN spin-spin-spin dynamics and flux/phasing (LO)

Tests of the results: dynamics

- Lorentz invariance (harmonic gauge)
- Set of conserved quantities
- Test-mass limit
- Equivalence of results with ADM/EFT

Tests of the results: flux

- Test-mass limit
- Source moments for boosted Kerr BH
- Equivalence with EFT?
The energy flux for quasi-circular spin-aligned orbits

\[ F = \frac{32\nu^2}{5G}c^5x^5 \left( 1 + \left( -\frac{1247}{336} - \frac{35}{12}\nu \right)x + \ldots \right) \]

\[ + \left( \left( -\frac{3839}{252} - 43\nu \right) S_\ell^2 + \left( -\frac{1375}{56} - 43\nu \right) \delta S_\ell \Sigma_\ell + \left( -\frac{227}{28} + \frac{3481\nu}{168} + 43\nu^2 \right) \Sigma_\ell^2 \right) x^6 \]

\[ + \left( \left( \frac{476645}{6804} + \frac{6172}{189}\nu - \frac{2810}{27}\nu^2 \right) S_\ell + \left( \frac{9535}{336} + \frac{1849}{126}\nu - \frac{1501}{36}\nu^2 \right) \frac{\delta m}{m} \Sigma_\ell \right) x^{7/2} \]

\[ + \left( -\frac{16}{3} S_\ell^3 + \frac{2}{3} \delta S_\ell^2 \Sigma_\ell + \left( \frac{9}{2} - \frac{56\nu}{3} \right) S_\ell \Sigma_\ell^2 + \left( \frac{35}{24} - 6\nu \right) \delta \Sigma_\ell^3 \right) x^{7/2} \]

\[ + \left( \left( -\frac{3485\pi}{96} + \frac{13879\pi}{72}\nu \right) S_\ell + \left( -\frac{7163\pi}{672} + \frac{130583\pi}{2016}\nu \right) \frac{\delta m}{m} \Sigma_\ell \right) x^4 \right) \]

**PN parameter:** \[ x \equiv \left( Gm\omega/c^3 \right)^{2/3} \]

**Masses:** \[ \nu = m_1 m_2/m^2 \quad \delta = (m_1 - m_2)/m \]

**Spins:** \[ S \sim S_1 + S_2 \quad \Sigma \sim S_2 - S_1 \]
Part IV

Result for the number of cycles

Phasing for circular orbits

\[ E(\omega), \mathcal{F}(\omega) \rightarrow \text{Balance equation} \]

\[ \mathcal{F} = -\frac{dE}{dt} \rightarrow \text{GW Phase} \]

Taylor T2

Number of cycles between f~10Hz and \( \omega = \omega_{\text{ISCO}} \) (\( x_{\text{ISCO}} = 1/6 \))

- Question of the convergence of the PN series
- Rough estimate of the importance of the new terms
- Approximant-dependent

<table>
<thead>
<tr>
<th>LIGO/Virgo</th>
<th>10(M_\odot ) + 1.4(M_\odot )</th>
<th>10(M_\odot ) + 10(M_\odot )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Newtonian</td>
<td>3558.9</td>
<td>598.8</td>
</tr>
<tr>
<td>1PN</td>
<td>212.4</td>
<td>59.1</td>
</tr>
<tr>
<td>1.5PN</td>
<td>(-180.9 + 114.0\chi_1 + 11.7\chi_2)</td>
<td>(-51.2 + 16.0\chi_1 + 16.0\chi_2)</td>
</tr>
<tr>
<td>2PN</td>
<td>(9.8 - 10.5\chi_1^2 - 2.9\chi_1\chi_2)</td>
<td>(4.0 - 1.1\chi_1^2 - 2.2\chi_1\chi_2 - 1.1\chi_2^2)</td>
</tr>
<tr>
<td>2.5PN</td>
<td>(-20.0 + 33.8\chi_1 + 2.9\chi_2)</td>
<td>(-7.1 + 5.7\chi_1 + 5.7\chi_2)</td>
</tr>
<tr>
<td>3PN</td>
<td>(2.3 - 13.2\chi_1 - 1.3\chi_2)</td>
<td>(2.2 - 2.6\chi_1 - 2.6\chi_2)</td>
</tr>
<tr>
<td></td>
<td>(-1.2\chi_1^2 - 0.2\chi_1\chi_2)</td>
<td>(-0.1\chi_1^2 - 0.2\chi_1\chi_2 - 0.1\chi_2^2)</td>
</tr>
<tr>
<td>3.5PN</td>
<td>(-1.8 + 11.1\chi_1 + 0.8\chi_2 + (\text{SS}))</td>
<td>(-0.8 + 1.7\chi_1 + 1.7\chi_2 + (\text{SS}))</td>
</tr>
<tr>
<td></td>
<td>(-0.7\chi_1^3 - 0.3\chi_1^2\chi_2)</td>
<td>(-0.05\chi_1^3 - 0.2\chi_1^2\chi_2 - 0.2\chi_1\chi_2^2 - 0.05\chi_2^3)</td>
</tr>
<tr>
<td>4PN</td>
<td>((\text{NS}) - 8.0\chi_1 - 0.7\chi_2 + (\text{SS}))</td>
<td>((\text{NS}) - 1.5\chi_1 - 1.5\chi_2 + (\text{SS}))</td>
</tr>
</tbody>
</table>
Part IV  Summary and Conclusion

Results

- 3.5PN spin-orbit dynamics, 4PN spin-orbit flux/phasing
- 3PN spin-spin dynamics and flux/phasing
- Lagrangian formalism for higher-order spin effects
- 3.5PN spin-cube dynamics and flux/phasing

Work in progress

- 3.5PN spin-orbit and 3PN spin-spin polarizations (or spherical modes)
- 3.5PN spin-spin tail effects
- 4PN non-spinning dynamics (and flux/phasing later)
- Spin effects at higher order: 4PN spin-spin, 4PN spin^4, 4.5PN spin-orbit
Part IV  
Illustration: phasing for the spin-aligned case

[Nitz&al 13]: matches between templates computed for aligned spins with fixed physical parameters

Agreement between approximants, at a given PN order:

Agreement between successive PN orders for each approximant:

More complete study needed to quantify this in terms of parameter estimation bias.
Part I

Effects of the spins: phasing and precession

- Affect the phasing (aligned)
- Orbital plane precession (misaligned)

Effects of the spins

\[ \ell = \hat{L}_N \]

Amplitude modulation for \( h_+, h_\times \)

\[ J \approx cte = L_N + S/c + \ldots \]

\[ \dot{S}_A = \Omega_A \times S_A \]

+ precessional phases

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Part II

Different post-Newtonian methods

- **DIRE** (Direct Integration of Relaxed Einstein equations): near-zone/far-zone split of retarded integrals [Will, Wiseman, ...]

- **Surface-integral** approach [Futamase, Itoh, ...]

- **Effective field theory** [Goldberger, Rothstein, ...]: diagrammatic computation of an effective action

- **ADM Hamiltonian formalism** [Schäfer, Damour, Jaranowski, ...]: field degrees of freedom integrated out, obtaining a reduced Hamiltonian

- **Harmonic coordinates, MPM algorithm + matching** [Blanchet, Damour, Iyer, ...]

Validation of results by different methods welcome!
Part II

Lagrangian formalism: conserved norm spin

The spin covector

Using the SSC to define a spin covector:

\[ S_\mu = \frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} \frac{p^\nu}{m} S^{\rho \sigma} \quad S_\mu p^\mu = 0 \]

Defining Euclidean norm spin vector

Defining a tetrad: \((e_0^\mu, e_I^\mu)\) with \(e_0^\mu \equiv u^\mu\)

Conserved norm vector:

\[ s^2 = (g^{\mu \nu} + u^\mu u^\nu) S_\mu S_\nu = \delta^{IJ} S_I S_J \]

Fixing the convention for the spatial part of the tetrad:

\[ \gamma_{ij} = g_{ij} + u_i u_j = \delta^{IJ} e_I e_J \]

Precession equation

\[ \frac{dS}{dt} = \Omega \times S \]

- Leading SO terms 1PN, leading SS terms 1.5PN
- Simplify the structure of equations (hereditary integrals)
- Important when applying the balance equation
Part II

Multipolar post-Newtonian wave generation formalism

$h\mapsto h$ near-zone

$r \ll \lambda$

Near-zone

$\lambda \ll r$

Radiation zone

$\mathcal{M}(h) = \mathcal{M}(\bar{h})$

$\mathcal{M}(h)$: multipolar expansion

$\bar{h}$: post-Newtonian (near-zone) expansion
Part II

Link between source and radiative moments

Outline

- MPM solution parametrized by linear solution source/gauge moments \( I_L, J_L, \ldots, Z_L \)
- Matching: source/gauge moments as spatial integrals \( \int d^3x (\ldots) \)
- Radiative coordinates and radiatives multipoles \( U_L, V_L \) describing waveform at infinity
- Finite part regularization: \( \int d^3x \to \text{FP}_{B=0} \int d^3x \left( \frac{|x|}{r_0} \right)^B \)

Result of MPM algorithm

Radiative quadrupole: \( U_{ij}(u) = I_{ij}^{(2)} + \frac{1}{c^5} \left[ I_{ai}^{(5)} I_{ja} + \ldots \right] \) Instantaneous

\[ + \frac{M}{c^3} \int_0^{+\infty} d\tau I_{ij}^{(4)} (u - \tau) \ln \left( \frac{\tau}{2\tau_0} \right) \] Tails

Hereditary contributions

\[ + \frac{1}{c^5} \int d\tau I_{ia}^{(3)} I_{aj}^{(3)} + \ldots \] Memory

\[ + \frac{M^2}{c^6} \int d\tau I_{ij}^{(5)} \left[ \ln^2 + \ln + \text{cte} \right] \] Tails of tails
### Part II

**Result of the matching equation and waveform**

#### Results of the matching

[Blanchet 98]

- Source and gauge moments expressed as integrals over the source:

\[
I_L = FP \int d^3 x \hat{x}_L \left( \sigma - \frac{1}{c^2} \Delta (V^2) + \frac{1}{c^4} (V \sigma_{ii} + V_i \partial_t \partial_i V) + \ldots \right)
\]

#### The near-zone PN metric from matching (4PN tails)

[Blanchet & Poujade 02]

#### Waveform and energy flux

[Thorne 80]

**Wave (Transverse-Traceless):**

\[
h_{ij}^{TT} = \frac{1}{c^2 R} \Lambda_{ij}^{TT} (N) \sum_{\ell \geq 2} \frac{1}{c^\ell} \left[ N U_L + \frac{1}{c} N \varepsilon V_L \right]
\]

**Emitted energy flux:**

\[
\mathcal{F} = \sum_{\ell \geq 2} \frac{1}{c^{2\ell+1}} \left[ \dot{U}_L \dot{U}_L + \frac{1}{c^2} \dot{V}_L \dot{V}_L \right]
\]
Part III

PN near-zone metric iteration

Matching for the near-zone metric

\[ \square h^{\mu\nu} = \tau^{\mu\nu} \quad \bar{h}^{\mu\nu} = \square_B^{-1} \tau^{\mu\nu} + h_{\text{tail}}^{\mu\nu} \]

- \( \square_B^{-1} \) PN-expanded inverse d'Alembertian with \( FP_{B=0} \) reg.
- \( h_{\text{tail}}^{\mu\nu} \) 4PN hereditary contribution (tails in RR)

Metric potentials

\[ \sigma \leftrightarrow T^{\mu\nu} \]

- \( g_{00} \rightarrow V/c^2, \hat{X}/c^6, \hat{T}/c^8 + \mathcal{O}(10) \)
- \( g_{0i} \rightarrow V_i/c^3, \hat{R}_i/c^5, \hat{Y}_i/c^7 + \mathcal{O}(9) \)
- \( g_{ij} \rightarrow \delta_{ij} V/c^2, \hat{W}_{ij}/c^4, \hat{Z}_{ij}/c^6 + \mathcal{O}(8) \)

Metric parametrized by potentials

Source equations for potentials

- \( V = \square_R^{-1} [-4\pi G \sigma] \)
- \( V_i = \square_R^{-1} [-4\pi G \sigma_i] \)
- \( \hat{W}_{ij} = \square_R^{-1} [-4\pi G (\sigma_{ij} - \delta_{ij} \sigma_{kk}) - \partial_i V \partial_j V] \)
- \( \hat{X} = \square_R^{-1} [-4\pi G V \sigma_{ii} + \hat{W}_{ij} \partial_{ij} V + \ldots] \)

Solution for the potentials

- Relies on explicit solutions e.g. \( \Delta^{-1}(1/r_1 r_2) = \ln(r_1 + r_2 + r_{12}) \)
- Regularization & distributional derivatives
- Potentials in all space or regularized

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Part II

Point particles with spin: earlier approaches

Papapetrou approach

[Papapetrou 51], generalization [Dixon]

Non-covariant approach: \( \mathcal{T}^{\mu\nu} \equiv \sqrt{-g}T^{\mu\nu} \), pole-dipole hypothesis:

\[
\int d^3x \mathcal{T}^{\mu\nu} \neq 0, \quad \int d^3x \delta x^\rho \mathcal{T}^{\mu\nu} \neq 0, \quad \delta x = x - \bar{x}
\]

Composite definitions for spin, linear momentum and mass:

\[
S^{\mu\nu} \equiv \int d^3x 2\delta x^{[\mu} \mathcal{T}^{\nu]} \nu, \quad p^\mu \equiv m u^\mu - u_\nu \frac{DS^{\mu\nu}}{d\tau}
\]

Evolution equations:

\[
\frac{Dp^\mu}{d\tau} = -\frac{1}{2} R_{\nu\rho\sigma} u^\nu S^{\rho\sigma} \quad \frac{DS^{\mu\nu}}{d\tau} = 2p^{[\mu}u^{\nu]} 
\]

Gravitational skeleton approach

[Mathisson 37], [Tulczyjew 59]

Ansatz on the stress-energy tensor

\[
T^{\mu\nu} = \int d\tau \left[ t^{\mu\nu} \delta + \nabla_\rho (t^{\mu\nu}\rho \delta) + \nabla_\rho \nabla_\sigma (t^{\mu\nu}\rho\sigma \delta) + \ldots \right]
\]

Method: unicity of the canonical decomposition

\[
\sum_k \int d\tau \nabla_{\alpha_1} \ldots \nabla_{\alpha_k} \left( A^{\alpha_1} \ldots \alpha_k \beta_1 \ldots \beta_m \delta \right)
\]

(for the \( \alpha_i \), symmetry and orthogonality to \( u^\mu \))

\[
\nabla_\nu T^{\mu\nu} = 0 \text{ rewritten in canonical form} \rightarrow \text{equations of evolution}
\]
Part II

Lagrangian formalism: SSC and definition of the mass

Spin supplementary condition

\[ S^\mu{}^\nu \text{ six degrees of freedom} \quad \rightarrow \quad \text{impose 3 conditions} \quad V_\mu S^\mu{}^\nu = 0 \]

Covariant SSC: \[ p_\mu S^\mu{}^\nu = 0 \]

Impose conservation of the SSC: relation \[ p^\mu \leftrightarrow u^\mu \]

Definition of the mass

\[ m^2 = -p_\mu p^\mu \quad \text{is not conserved at order SS 3PN} \]

Alternative definition (not general at all PN orders):

\[ \tilde{m} \equiv -p_\mu u^\mu - \frac{1}{6} J^{\lambda\nu\rho\sigma} R_{\lambda\nu\rho\sigma} , \quad \frac{\mathrm{d}\tilde{m}}{\mathrm{d}\tau} = \mathcal{O}(S^3/c^9) \]
Illustration: comparison of PN/NR precession

[Ossokine & al 15]: comparison of PN (harmonic) and NR (Spec) precession

FIG. 1. Precession cones of the six primary precessing simulations considered here, as computed by NR and PN. Shown are comparison of PN (harmonic) and NR (Spec) precession.
When applied to the NR simulation, the TaylorT4 approximation can be used to implement the PN expressions for the spin precession frequencies through next-to-next-to-leading order (corresponding to $3.5 \, \text{PN}$). Similarly, the spin precession frequencies might be estimated at different times for varying PN orders, which allows for an investigation of the convergence properties of the PN series and enables more accurate post-Newtonian information. It is also instructive to examine the behavior of the spin precession angles $l_{\text{PN}}$, $l_{\text{NR}}$, $S_{\text{PN}}$, and $S_{\text{NR}}$ at $3.5 \, \text{PN}$ order. Let us now investigate the opposite case: when we match PN with different orders of the orbital frequency evolution. This comparison is illustrated in Fig. 12, where the top panel shows the precession angle $\angle L$ at $3.5 \, \text{PN}$ order. The bottom panel displays the evolution of $\angle L$ at $3.5 \, \text{PN}$ order. The results are shown in Fig. 14, where the angles are compared for varying PN orders.

The angles $\langle l_{\text{PN}}, l_{\text{NR}} \rangle$ and $\langle S_{\text{PN}}, S_{\text{NR}} \rangle$ at a specific time are shown in the right panel of the figure, highlighting the improvement in accuracy as the PN order increases. These results suggest that convergence of the orbital evolution (and TaylorT4 energy-balance prescription) to $3 \, \text{PN}$ order is monotonic, with a large improvement at $1.5 \, \text{PN}$ order, associated with the leading order spin-orbit contribution. The phase differences then spike at $2$ and $2.5 \, \text{PN}$ orders and then decrease as the PN order increases.

For the anti-aligned cases, the picture is similar to pre-aligned spin and anti-aligned spin binaries listed in Table 12 fits. In the top panel of Fig. 14, the results are shown for different cases, where the parameter space is warranted. Further investigation of these results is required to understand the behavior of the spin cases. Each data point shown is averaged over PN-NR varying PN order.

Reference: [Ossokine & al 15]: comparison of PN (harmonic) and NR (Spec) precession.
**Comparison**

- Satisfying agreement for the precession (even if gauge-dependent quantities)

- Convergence less clear for the orbital phase…

---

**Ossokine & al 15:** comparison of PN (SpinTaylorT4) and NR (Spec) phasing

![Graphs showing comparison of PN and NR phasing](image-url)

**Part IV**

Illustration: comparison of PN/NR phasing
Part I

The spin of black holes: observations

The Kerr black hole

Most general stationary, axisymmetric vacuum solution to Einstein equations: the rotating Kerr black hole

Dimensionless Kerr parameter: $a \equiv \frac{cJ}{Gm^2}$ (1 for maximally rotating black hole)

X-Ray spectroscopy of accretion disks

Example for stellar mass black holes: [Gou & al 11] $a > 0.95$ for Cygnus X-1

Summary for SMBH [Reynolds 13]
The spin of a merger remnant

Numerical relativity results:

Spin of the remnant for nonspinning black holes [e.g. Gonzalez&al 07]:

Effective formulas for spinning BH binaries [e.g. Rezzolla&al 08]

Link with astrophysics

Inverse problem: what will the measured distribution of spins tell us about their environment, and about the growth history (accretion or merger) of SMBH?
Part II  

Post-Newtonian expansion

PN conventions

- Slowly-varying, weakly-gravitating regime: \(1\)PN \(\sim Gm/rc^2 \sim v^2/c^2\)
- Convention: \(S = cJ = Gm^2a\), of Newtonian order for an extremal BH.

Spin corrections to the equations of motion

\[
\frac{dv}{dt} = A_N + \frac{1}{c^2} A_{1PN} + \frac{1}{c^4} A_{2PN} + \frac{1}{c^5} A_{2.5PN}^{RR} + \frac{1}{c^6} A_{3PN} + \frac{1}{c^7} A_{3.5PN}^{RR} \\
+ \frac{1}{c^3} A_{1.5PN}^{SO} + \frac{1}{c^5} A_{2.5PN}^{SO} + \frac{1}{c^7} A_{3.5PN}^{SO} + O \left( \frac{1}{c^8} \right)
\]

Spin corrections to the energy flux

(spinspin terms not shown)

\[
\mathcal{F} = F_N + \frac{1}{c^2} F_{1PN} + \frac{1}{c^3} F_{1.5PN}^{tails} + \frac{1}{c^4} F_{2PN} + \frac{1}{c^5} F_{2.5PN}^{tails} + \frac{1}{c^6} F_{3PN} + \frac{1}{c^7} F_{3.5PN}^{tails} \\
+ \frac{1}{c^3} F_{1.5PN}^{SO} + \frac{1}{c^5} F_{2.5PN}^{SO} + \frac{1}{c^6} F_{3PN}^{SO-tails} + \frac{1}{c^7} F_{3.5PN}^{SO} + \frac{1}{c^8} F_{4PN}^{SO-tails} + O \left( \frac{1}{c^9} \right)
\]

ADM Hamiltonian derived by [Hartung-Steinhoff 2011]

Addressed in this work
Part II

Latest post-Newtonian results for spin effects

ADM Hamiltonian results:

- Next-to-leading order Hamiltonian, S-O [Damour, Jaranowski, Schäfer 07]
- Next-to-leading order Hamiltonian, S1-S2 [Steinhoff, Hergt, Schäfer 07]
- Next-to-leading order Hamiltonian, S^2 [Hergt, Steinhoff, Schäfer 10]
- Next-to-next-to-leading order Hamiltonian, S-O and S1-S2 [Hartung&Steinhoff 11]

EFT results:

- Next-to-leading order, S-O [Porto 10]
- Next-to-leading order, S1-S2 and S^2 [Porto&Rothstein 10, Levi 08, Levi 10]
- Next-to-next-to-leading order S1-S2 [Porto&Rothstein 11, Levi 11]

(And so far incomplete results for the waveform and flux)

Harmonic coordinates results:

- Next-to-leading order, S-O (EOM and flux) [Faye, Blanchet, Buonanno 06]
- Next-to-leading order, S-O (full waveform) [Arun&al 08]
- Leading order, S1-S2 and S^2 (full waveform) [Buonanno, Faye, Hinderer 12]
- Next-to-next-to-leading order S-O (EOM and flux) [this work]
Part II
Multipolar post-newtonian wave generation formalism

Outline

- Iteration of \( h = \mathcal{L}^{-1}(h) \) outside the source starting with a linear solution parametrized by source and gauge moments \( I_L, J_L, \ldots, Z_L \)

- Existence of a matching region for a PN source matching of asymptotic expansions
  \( \rightarrow I_L, \ldots, Z_L \) as integrals over the source
  \( \rightarrow \) consistent PN iteration in the near zone

- Radiative coordinates and radiatives multipoles \( U_L, V_L \) describing waveform at infinity

- Alternative parametrization in terms of only two sets of canonical moments \( M_L, S_L \)
  \( \rightarrow \) relation found by a gauge transformation

Finite part regularization

\[
\int d^3x \rightarrow FP_{B=0} \int d^3x \left( \frac{|x|}{r_0} \right)^B
\]
Part II

UV regularization

Hadamard regularization

- Regularized value of singular functions:
  \[ F(x) = \sum_{p_0 \leq p \leq N} r_1^p f_p(n_1) + o(r_1^N), \quad (F)_1 = \langle f_0(n_1) \rangle \]

- Non-distributive: \( F\delta_1 \neq (F)_1 \delta_1, \quad (FG)_1 \neq (F)_1 (G)_1 \)

- Prescription for distributional derivatives (not unique, no Leibniz rule)

- Regularization of integrals: removal of the diverging part \( \text{Pf}_{s_1,s_2} \int d^3xF(x) \)

- Apparition of ambiguities at the 3PN NS order

Dimensional regularization

- \( d \to 3 + \varepsilon \) and analytical continuation in \( \varepsilon \)

- Structure: \( F^{(d)}(x) = \sum_{p_0 \leq p \leq N} r_1^{p+q\varepsilon} f_p^{(\varepsilon)}(n_1) + o(r_1^N), \quad f_{p}(n_1) = \sum_{q_0 \leq q \leq q_1} f_{p,q}^{(0)}(n_1) \)

- Distributive, well-defined distributional prescription, regular integrals

- In practice: ‘pure Hadamard-Schwartz’ supplemented by dimreg

Determined the 3PN ambiguities
Part III

Results for the metric and applications

Metric in the whole near-zone

\[(g_{00})_S \rightarrow \mathcal{O}(7)\]
\[(g_{0i})_S \rightarrow \mathcal{O}(6)\]
\[(g_{ij})_S \rightarrow \mathcal{O}(7)\]

Can be used for:

- Building approximate solutions by asymptotic matching to a perturbed black hole [Gallouin&al 12]
- Simulating a circumbinary MHD disk in a PN-approximated spacetime [Noble&al 09]
- Building realistic initial conditions for NR using PN information [Kelly&al 09]

Regularized metric

\[(g_{00}^S)_1 \rightarrow \mathcal{O}(9)\]
\[(g_{0i}^S)_1 \rightarrow \mathcal{O}(8)\]
\[(g_{ij}^S)_1 \rightarrow \mathcal{O}(7)\]

- Used for the first law of binary black holes [Blanchet&al 12]

With EOM:

Allows computation of the emitted waveform and energy flux
Part III  Computation of the potentials I

Compact-support terms

Dirac-delta terms (stress-energy tensor or distributional contributions), treated with pHS:

\[ \int d^3x F(x) \delta_1 = (F)_1 \]

‘Easy’ non-compact-support terms

Particular solution:

\[ \Delta g = \frac{1}{r_1 r_2}, \]
\[ g \equiv \ln(r_1 + r_2 + r_{12}) \]

Quadratic terms with lowest-order potentials \( V, V_i \) can be readily integrated:

\[ \Delta^{-1} \left[ \partial_i \left( \frac{1}{r_1} \right) \partial_j \left( \frac{1}{r_2} \right) \right] = -\partial_i^1 \partial_{jk}^2 g \]

‘Difficult’ non-compact-support terms

Only the regularized potential is evaluated, using generic formulas:

\[ P(x) = -\frac{1}{4\pi} \text{Pf} s_1, s_2 \int \frac{d^3x'}{|x - x'|} F(x') \]

\[ (P)_1 = -\frac{1}{4\pi} \text{Pf} s_1, s_2 \int \frac{d^3x}{r_1} F(x) \left[ \ln \left( \frac{r_1'}{s_1} \right) - 1 \right] \left( r_1^2 F \right)_1 \]

Regularization constants

s_1, s_2, r_1', r_2'
Part III
Computation of the potentials II

‘Difficult’ non-compact-support terms

$$(\partial_{jk} \hat{Y}_i^{NS})_1$$

No apparition of reg. constants in the spin part of potentials

Apparition of gauge constants in one no-spin potential: dimreg computation

Dimreg contributions

$$D(\partial_{ij} P)(1) \equiv (\partial_{ij} P^{(d)})(y_1) - (\partial_{ij} P)_1$$

Result for the pole:

$$D(\partial_{jk} \hat{Y}_i)(1) = \frac{1}{\varepsilon} \frac{G^3 m_1^2 m_2}{252} v_{12} \partial_{ijkl}^{(d)} \left( \frac{1}{r_{12}} \right) + O(\varepsilon^0)$$

$$(\partial_{j[k} \hat{Y}_i^{NS})_1$$

Cancellation of all dimreg contributions
Part IV

Overview for the waveform and flux and 3.5PN order

- Equations of motion and precession
- Metric potentials computed in all space

Matching: expressed as integrals over the source

\[ I_L = \text{FP} \int d^3x \hat{x}_L \left( \sigma - \Delta (V^2) / c^2 + \ldots \right) \]

Radiative moments

\[ U_L, V_L \]

\[ h_{ij}^{TT} = \frac{1}{c^2 R} \Lambda_{ij}^{TT}(N) \sum_{\ell \geq 2} \frac{1}{c^\ell} \left[ NU_L + \frac{1}{c} N \varepsilon V_L \right] \]

\[ F = \sum_{\ell \geq 2} \frac{1}{c^{2\ell+1}} \left[ \dot{U}_L \dot{U}_L + \frac{1}{c^2} \dot{V}_L \dot{V}_L \right] \]

- At 3.5PN order, only leading order instantaneous contributions intervene (with leading tail terms at 3PN):
  \[ U_L = I_L^{(1)}, V_L = J_L^{(1)} \]
- Computation of the source moments and their derivatives using EOM and metric
Part III

Results for quasi-circular orbits

Equations of motion

Corrections in Kepler’s law:

\[ x \equiv \left( \frac{Gm\omega}{c^3} \right)^{2/3} \]

\[
\frac{Gm}{rc^2} = x \left\{ 1 + x \left( 1 - \frac{1}{3} \nu \right) + \ldots \right. \\
+ \left. \frac{x^{7/2}}{Gm^2} \left[ \left( \frac{5}{12} - \frac{127}{12} \nu - 6\nu^2 \right) S_\ell + \frac{\delta m}{m} \left( 3 - \frac{61}{6} \nu - \frac{8}{3} \nu^2 \right) \Sigma_\ell \right] \right\} + O(8).
\]

Conserved quantities

Corrections in the orbital energy:

\[
E = -\frac{mc^2 x}{2} \left\{ 1 + x \left( -\frac{3}{4} - \frac{1}{12} \nu \right) + \ldots \right. \\
+ \left. \frac{x^{7/2}}{Gm^2} \left[ \left( \frac{135}{4} - \frac{367}{4} \nu + \frac{29}{12} \nu^2 \right) S_\ell + \frac{\delta m}{m} \left( \frac{27}{4} - 39\nu + \frac{5}{4} \nu^2 \right) \Sigma_\ell \right] \right\} + O(8).
\]
Part IV  From the energy and flux to the phase

Spin contributions in the balance equation

\[ \mathcal{F} = -\frac{dE}{dt} \quad \rightarrow \quad \dot{x} \frac{dE}{dx} + \dot{S} \frac{dE}{dS} = -\mathcal{F} \]

Post-Newtonian orders: control of the evolution of the spins?

\[ \mathcal{O}(5)\mathcal{(O}(0) + \cdots + \mathcal{O}(7)) + \dot{S}_\ell(\mathcal{O}(3) + \cdots + \mathcal{O}(7)) = \mathcal{O}(5)(\mathcal{O}(0) + \cdots + \mathcal{O}(7)) \]

Secular spin variables at linear order in spin: \( \dot{S}_\ell = \mathcal{O}(S^2) \) since \( \dot{S} = \Omega \times S \), \( \Omega \propto \ell \)

Illustration of the computation of the phase

- Taylor T2: solve analytically after PN-expanding the system

\[
\begin{align*}
\frac{d\phi}{dx} &= -\frac{c^3}{Gm} x^{3/2} \frac{dE/dx}{\mathcal{F}(x)} \\
\frac{dt}{dx} &= -\frac{dE/dx}{\mathcal{F}(x)}
\end{align*}
\]

- Taylor T1: solve numerically without re-expanding the system

\[
\begin{align*}
\frac{dx}{dt} &= -\frac{\mathcal{F}}{dE/dx} \\
\frac{d\phi}{dt} &= \frac{c^3}{Gm} x^{3/2}
\end{align*}
\]
Part IV

Result for the number of cycles

**Taylor T2**

Number of cycles between $f \sim 10\text{Hz}$ and $\omega = \omega_{\text{ISCO}}$ ($x_{\text{ISCO}} = 1/6$)

<table>
<thead>
<tr>
<th></th>
<th>$1.4M_\odot + 1.4M_\odot$</th>
<th>$10M_\odot + 1.4M_\odot$</th>
<th>$10M_\odot + 10M_\odot$</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>15952.6</td>
<td>3558.9</td>
<td>598.8</td>
</tr>
<tr>
<td>1PN</td>
<td>439.5</td>
<td>212.4</td>
<td>59.1</td>
</tr>
<tr>
<td>1.5PN</td>
<td>$-210.3 + 65.6\kappa_1\chi_1 + 65.6\kappa_2\chi_2$</td>
<td>$-180.9 + 114.0\kappa_1\chi_1 + 11.7\kappa_2\chi_2$</td>
<td>$-51.2 + 16.0\kappa_1\chi_1 + 16.0\kappa_2\chi_2$</td>
</tr>
<tr>
<td>2PN</td>
<td>9.9</td>
<td>9.8</td>
<td>4.0</td>
</tr>
<tr>
<td>2.5PN</td>
<td>$-11.7 + 9.3\kappa_1\chi_1 + 9.3\kappa_2\chi_2$</td>
<td>$-20.0 + 33.8\kappa_1\chi_1 + 2.9\kappa_2\chi_2$</td>
<td>$-7.1 + 5.7\kappa_1\chi_1 + 5.7\kappa_2\chi_2$</td>
</tr>
<tr>
<td>3PN</td>
<td>$2.6 - 3.2\kappa_1\chi_1 - 3.2\kappa_2\chi_2$</td>
<td>$2.3 - 13.2\kappa_1\chi_1 - 1.3\kappa_2\chi_2$</td>
<td>$2.2 - 2.6\kappa_1\chi_1 - 2.6\kappa_2\chi_2$</td>
</tr>
<tr>
<td>3.5PN</td>
<td>$-0.9 + 1.9\kappa_1\chi_1 + 1.9\kappa_2\chi_2$</td>
<td>$-1.8 + 11.1\kappa_1\chi_1 + 0.8\kappa_2\chi_2$</td>
<td>$-0.8 + 1.7\kappa_1\chi_1 + 1.7\kappa_2\chi_2$</td>
</tr>
<tr>
<td>4PN</td>
<td>(NS) $-1.5\kappa_1\chi_1 - 1.5\kappa_2\chi_2$</td>
<td>(NS) $-8.0\kappa_1\chi_1 - 0.7\kappa_2\chi_2$</td>
<td>(NS) $-1.5\kappa_1\chi_1 - 1.5\kappa_2\chi_2$</td>
</tr>
</tbody>
</table>

$\kappa_i, \chi_i$ parameters for the orientation and magnitude of the spins

**Taylor T1**

Aligned spins, 0.1 for neutron stars and 1 for black holes

<table>
<thead>
<tr>
<th></th>
<th>$1.4M_\odot + 1.4M_\odot$</th>
<th>$10M_\odot + 1.4M_\odot$</th>
<th>$10M_\odot + 10M_\odot$</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>16028.2</td>
<td>3575.8</td>
<td>601.6</td>
</tr>
<tr>
<td>1PN</td>
<td>474.4</td>
<td>248.7</td>
<td>75.8</td>
</tr>
<tr>
<td>1.5PN</td>
<td>$-237.1 + (13.7)_S$</td>
<td>$-214.9 + (122.5)_S$</td>
<td>$-67.2 + (35.0)_S$</td>
</tr>
<tr>
<td>2PN</td>
<td>$-18.5$</td>
<td>$-182.$</td>
<td>$-8.0$</td>
</tr>
<tr>
<td>2.5PN</td>
<td>$20.8 + (0.6)_S$</td>
<td>$33.6 + (16.2)_S$</td>
<td>$16.6 + (3.9)_S$</td>
</tr>
<tr>
<td>3PN</td>
<td>$-10 + (0.2)_S$</td>
<td>$-30.3 + (4.6)_S$</td>
<td>$-11.6 + (1.8)_S$</td>
</tr>
<tr>
<td>3.5PN</td>
<td>$-0.1 + (-0.01)_S$</td>
<td>$2.7 + (1.3)_S$</td>
<td>$-0.2 + (-0.3)_S$</td>
</tr>
<tr>
<td>4PN</td>
<td>(NS) $+ (-0.005)_S$</td>
<td>(NS) $+ (0.4)_S$</td>
<td>(NS) $+ (-0.1)_S$</td>
</tr>
</tbody>
</table>
Observations of rotation of neutron stars:

Two main pulsar populations:
- Young, normal pulsars
- Recycled pulsars: $P \sim$ few milliseconds

Dimensionless Kerr parameter:

Fastest known pulsar: J1748-2446, 716 Hz
Order-of-magnitude estimate (I not known):
$$a \sim 0.4$$
Typical value in binaries:
$$a \sim 0.1$$
Comparison NR/PN for the 22 mode

FIG. 21: TaylorT4 amplitude comparison for different PN orders. Shown is the relative difference in gravitational wave amplitude between TaylorT4 and numerical Y_{22} waveforms as a function of time. Matching is performed at \( m \omega = 0 \).

Comparing different post-Newtonian approximants

The previous section presented detailed comparisons of our numerical waveforms with four different post-Newtonian approximants. We now turn our attention to some comparisons between these approximants. In this section we also explore further how the post-Newtonian order influences agreement between numerical and post-Newtonian waveforms.

Figure 22 presents phase differences as a function of time for all four PN approximants we consider here and for different PN orders. The post-Newtonian and numerical waveforms are matched at \( m \omega = 0.04 \), about 9 cycles after the beginning of the numerical waveform, and about 21 cycles before its end. We find that some PN approximants at some particular orders agree exceedingly well with the numerical results. The best match is easily TaylorT4 at 3.5PN order, and the next best is 3.5/3.0, respectively; the labels along the top horizontal axes give the number of gravitational-wave cycles before \( m \omega = 0 \).

To get the complete waveform to 3PN order, only the (2,2) mode must be known to 3PN order; other modes must be known to smaller PN orders. For an equal mass, non-spinning binary, all modes except the (3,2) mode are currently known to sufficient order to get a complete 3PN waveform [62].

[Boyle&al 07]

FIG. 22: Phase comparison for different PN approximants at different PN orders, matched at \( m \omega = 0.04 \). Shown is the difference in gravitational wave phase between each post-Newtonian approximant and the numerical \( Y_{22} \) waveforms as a function of time. The vertical brown line indicates when the numerical simulation reaches \( m \omega = 0 \).

FIG. 23: Same as Fig. 22, but showing only the last stage of the inspiral. The horizontal axis ends at the estimated time of merger, \( (t-r^*)_{CAH} = 3950 \) m, cf. Section II G. The top and bottom panels use different vertical scales.
Transitional precession (20+5)M [BCV 02]: regime where S and L almost cancel, and direction of J changes rapidly

Maximal kick: “Hangup” configurations [Lousto&Zlochower 12]