A unifying description of Dark Energy & Modified Gravity

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Outline

1. Introduction
2. Main formalism
3. Horndeski’s theories
4. Theories beyond Horndeski
5. Link with cosmological observations

In collaboration with J. Gleyzes, F. Piazza & F. Vernizzi:
- General formalism 1304.4840, 1411.3712 (review + a few extensions)
- Theories beyond Horndeski: 1404.6495, 1408.1952
Introduction & motivations

• Plethora of dark energy models:
  – Dynamical dark energy: quintessence, K-essence
  – Modified gravity

• Large amount of data from future large scale cosmological surveys (LSST, Euclid, …)

• Goal: effective description as a bridge between models and observations.
Effective approach
Observational constraints
Theories
Effective approach
Observational constraints
Introduction & motivations

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• Assumptions:
  – Single scalar field models
  – All matter fields minimally coupled to the same metric $g_{\mu\nu}$
• The scalar field defines a **preferred slicing**
  Constant time hypersurfaces = uniform field hypersurfaces

\[
\phi = \phi_1 \\
\phi = \phi_2 \\
\phi = \phi_3
\]

• **ADM decomposition** based on this preferred slicing
Uniform scalar field slicing

- **Basic ingredients**
  - **Unit vector normal** to the hypersurfaces
    \[ n_\mu = - \frac{1}{\sqrt{-X}} \nabla_\mu \phi , \quad X \equiv g^{\rho \sigma} \nabla_\rho \phi \nabla_\sigma \phi \]
  - **Projection** on the hypersurfaces:
    \[ h_{\mu \nu} = g_{\mu \nu} + n_\mu n_\nu \]
  - **Intrinsic curvature** tensor \( (3)R_{\mu \nu} \)
  - **Extrinsic curvature** tensor
    \[ K_{\mu \nu} = h_{\mu \sigma} \nabla^\sigma n_\nu \quad K = \nabla_\mu n^\mu \]
ADM formulation

• ADM decomposition of spacetime

\[ ds^2 = -N^2 dt^2 + h_{ij} (dx^i + N^i dt)(dx^j + N^j dt) \]

Inverse metric

\[ g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^j/N^2 \\ N^i/N^2 & h^{ij} - N^i N^j/N^2 \end{pmatrix} \]

Hence

\[ X \equiv g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi = g^{00} \phi^2 = -\frac{\dot{\phi}^2(t)}{N^2} \]

• Lagrangians of the form

\[ S_g = \int d^4x N \sqrt{h} L(N, K_{ij}, R_{ij}, h_{ij}, D_i; t) \]
Example: GR + quintessence

• Consider a quintessence model

\[ S = \int d^4 x \sqrt{-g} \left[ \frac{M_P^2}{2} (^{(4)}R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)) \right] \]

• For the Einstein-Hilbert term, one can use

\[ ^{(4)}R = K_{\mu\nu} K^{\mu\nu} - K^2 + R + 2 \nabla_\mu (K n^\mu - n^\rho \nabla_\rho n^\mu) \]

• In the uniform \( \phi \) slicing, this leads to the Lagrangian

\[ L = \frac{M_{P1}^2}{2} \left[ K_{ij} K^{ij} - K^2 + R \right] + \frac{\dot{\phi}^2(t)}{2N^2} - V(\phi(t)) \]
Homogeneous evolution

• FLRW metric: \[ ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij} dx^i dx^j \]

• Extrinsic curvature: \[ K^i_j = \frac{\dot{a}}{Na} \delta^i_j \equiv H \delta^i_j \]

• Homogeneous Lagrangian

\[ \bar{L}(a, \dot{a}, \bar{N}) \equiv L \left[ K^i_j = \frac{\dot{a}}{\bar{N}a} \delta^i_j, R^i_j = 0, N = \bar{N}(t) \right] \]

• One can include **matter** by adding the Lagrangian for matter, minimally coupled to the metric.
Friedmann equations

• Variation of the action

\[ \bar{S}_g = \int dt \, d^3 x \, \bar{N} a^3 \, \bar{L}(a, \dot{a}, \bar{N}) \]

\[ \delta \bar{S}_g = \int dt \, d^3 x \left\{ a^3 \left( \bar{L} + \bar{N} L_N - 3H\mathcal{F} \right) \delta \bar{N} + 3a^2 \bar{N} \left( \bar{L} - 3H\mathcal{F} - \frac{\dot{\mathcal{F}}}{\bar{N}} \right) \delta a \right\} \]

with

\[ \left( \frac{\partial L}{\partial K_{ij}} \right)_{bgd} \equiv \mathcal{F} \bar{h}^{ij} \]

\[ \delta \bar{S}_m = \int dt \, d^3 x \, \bar{N} a^3 \left( -\rho_m \frac{\delta \bar{N}}{\bar{N}} + 3p_m \frac{\delta a}{a} \right) \left[ \delta S_m = \frac{1}{2} \int d^4 x \sqrt{-g} T^{\mu\nu} \, \delta g_{\mu\nu} \right] \]

• Friedmann equations

\[ \bar{L} + \bar{N} L_N - 3H\mathcal{F} = \rho_m \]

\[ \bar{L} - 3H\mathcal{F} - \frac{\dot{\mathcal{F}}}{\bar{N}} = -p_m \]
Linear perturbations

- Perturbations

\[ \delta N \equiv N - \bar{N}, \quad \delta K^j_i \equiv K^j_i - H \delta_i^j \]

- Expand the Lagrangian up to quadratic order:

\[
L(N, K^i_j, R^i_j, \ldots) = \bar{L} + L_N \delta N + \frac{\partial L}{\partial K^i_j} \delta K^i_j + \frac{\partial L}{\partial R^i_j} \delta R^i_j + L^{(2)} + \ldots
\]

with

\[
L^{(2)} = \frac{1}{2} L_{NN} \delta N^2 + \frac{1}{2} \frac{\partial^2 L}{\partial K^i_j \partial K^k_l} \delta K^i_j \delta K^k_l + \frac{1}{2} \frac{\partial^2 L}{\partial R^i_j \partial R^k_l} \delta R^i_j \delta R^k_l
\]

\[+ \frac{\partial^2 L}{\partial K^i_j \partial R^l_k} \delta K^i_j \delta R^l_k + \frac{\partial^2 L}{\partial N \partial K^i_j} \delta N \delta K^i_j + \frac{\partial^2 L}{\partial N \partial R^i_j} \delta N \delta R^i_j + \ldots \]
Linear perturbations

• The coefficients of the quadratic Lagrangian are evaluated on the homogeneous background, e.g.

\[ \frac{\partial^2 L}{\partial K^j_i \partial K^l_k} \equiv \hat{A}_K \delta^i_j \delta^k_l + A_K (\delta^i_j \delta^k_l + \delta^i_k \delta^j_l) \]

\[ \frac{\partial^2 L}{\partial R^j_i \partial R^l_k} \rightarrow (\hat{A}_R, A_R) \quad \frac{\partial^2 L}{\partial K^j_i \partial R^l_k} \rightarrow (\hat{C}, C) \quad \ldots \]

• For simplicity, we assume the three conditions

\[ \hat{A}_K + 2A_K = 0, \quad \hat{C} + \frac{1}{2} C = 0, \quad 4\hat{A}_R + 3A_R = 0 \]

so that the EOM is 2\textsuperscript{nd} order in spatial gradients.
Linear perturbations

- Action at quadratic order depends only on **five time-dependent functions**

\[ S^{(2)} = \int dx^3 dt a^3 \frac{M^2}{2} \left[ \delta K^i_j \delta K^j_i - \delta K^2 + \alpha_K H^2 \delta N^2 + 4 \alpha_B H \delta K \delta N \right. 
\]
\[ + (1 + \alpha_T) \delta_2 \left( \frac{\sqrt{h}}{a^3} R \right) + (1 + \alpha_H) R \delta N \]

where the alpha’s [Bellini & Sawicki’s notation] are explicitly given in terms of the derivatives of the Lagrangian.

- GR: \( M = M_P, \) \( \alpha_i = 0 \)
- Quintessence, K-essence: \( \alpha_K \neq 0 \)
- Brans-Dicke, F(R): \( M = M(t) \)
- Kinetic braiding: \( \alpha_B \neq 0 \)
- Horndeski (\( \alpha_H = 0 \)) and beyond Horndeski (\( \alpha_H \neq 0 \))
Linear degrees of freedom

- Scalar & tensor perts: \( h_{ij} = a^2(t) e^{2\zeta} (\delta_{ij} + \gamma_{ij}^{TT}) \)

- Quadratic action for the true degrees of freedom:

\[
S^{(2)} = \frac{1}{2} \int d^3x \, dt \, a^3 \left[ \mathcal{L}_{\zeta \dot{\zeta} \dot{\zeta}} + \mathcal{L}_{\partial \zeta \partial \zeta} \frac{1}{a^2} \mathcal{C} + \frac{M^2}{4} \dot{\gamma}_{ij}^2 - \frac{M^2}{4} (1 + \alpha_T) \frac{(\partial_k \gamma_{ij})^2}{a^2} \right]
\]

\[
\mathcal{L}_{\zeta \dot{\zeta}} \equiv M^2 \frac{\alpha_K + 6\alpha_B}{(1 + \alpha_B)^2}, \quad \mathcal{L}_{\partial \zeta \partial \zeta} \equiv 2M^2 \left\{ 1 + \alpha_T - \frac{1 + \alpha_H}{1 + \alpha_B} \left( 1 + \alpha_M - \frac{\dot{H}}{H^2} \right) - \frac{1}{H} \frac{d}{dt} \left( \frac{1 + \alpha_H}{1 + \alpha_B} \right) \right\}
\]

- Stability
  - No ghost: \( \mathcal{L}_{\zeta \dot{\zeta}} > 0, \quad M^2 > 0 \)
  
  - No gradient instability: \( c_s^2 \equiv -\frac{\mathcal{L}_{\partial \zeta \partial \zeta}}{\mathcal{L}_{\zeta \dot{\zeta}}} > 0, \quad c_T^2 \equiv 1 + \alpha_T > 0 \)
Horndeski theories

Horndeski 74; Nicolis et al. 08; Deffayet et al. 09 & 11

• Most general scalar-tensor action leading to at most second order equations of motion for the scalar field and metric [Horndeski ’74].

Combination of the following four Lagrangians

\[ L_2^H = G_2(\phi, X) \]
\[ L_3^H = G_3(\phi, X) \Box \phi \]
\[ L_4^H = G_4(\phi, X)^{(4)}R - 2G_4X(\phi, X)(\Box \phi^2 - \phi^{\mu\nu} \phi_{\mu\nu}) \]
\[ L_5^H = G_5(\phi, X)^{(4)}G_{\mu\nu} \phi^{\mu\nu} + \frac{1}{3} G_5X(\phi, X)(\Box \phi^3 - 3 \Box \phi \phi_{\mu\nu} \phi^{\mu\nu} + 2 \phi_{\mu\nu} \phi^{\mu\sigma} \phi^{\nu}_{\sigma}) \]

• ADM formulation ?
**Horndeski into ADM form**

**Goal:** translate the Lagrangian that depend on scalar field derivatives into an ADM expression.

\[ \phi_\mu = -\sqrt{-X} n_\mu \]
\[ \phi_{\mu\nu} = -\sqrt{-X} (K_{\mu\nu} - n_\mu a_\nu - n_\nu a_\mu) - \frac{1}{2\sqrt{-X}} n_\mu n_\nu n^\lambda \nabla_\lambda X \]

with \( a^\mu \equiv n^\lambda \nabla_\lambda n^\mu \)

Gauss-Codazzi relations are also useful.

\[ R_{\mu\nu} = (^{(4)}R_{\mu\nu})_\parallel + (n^\sigma n^\rho (^{(4)}R_{\mu\sigma\nu\rho})_\parallel) - KK_{\mu\nu} + K_{\mu\sigma}K^\sigma_\nu \]
\[ R = (^{(4)}R + K^2 - K_{\mu\nu}K^{\mu\nu} - 2\nabla_\mu (Kn^\mu - a^\mu) \]
Horndeski into ADM form

Combination of the Lagrangians

\[ L_2 = A_2 \]
\[ L_3 = A_3 \ K \]
\[ L_4 = A_4 \ (K^2 - K_{ij} K^{ij}) + B_4 \ R \]
\[ L_5 = A_5 \ (K^3 - 3 K K_{ij} K^{ij} + 2 K_{ij} K^{ik} K^{j}_k) + B_5 \ K^{ij} \ [R_{ij} - h_{ij} R / 2] \]

with

\[ A_2 = G_2 - \sqrt{-X} \int \frac{G_{3\phi}}{2\sqrt{-X}} \ dX, \]
\[ A_3 = - \int G_3 X \sqrt{-X} \ dX - 2\sqrt{-X} G_{4\phi}, \]
\[ A_4 = - G_4 + 2 X G_{4,X} + \frac{X}{2} G_{5,\phi}, \]
\[ A_5 = - \frac{1}{3} (-X)^{\frac{3}{2}} G_{5,X} \]
\[ B_4 = G_4 + \sqrt{-X} \int \frac{G_{5\phi}}{4\sqrt{-X}} \ dX, \]
\[ B_5 = - \int G_{5,X} \sqrt{-X} \ dX \]
\[ A_4 = - B_4 + 2 X B_{4,X} \]
\[ A_5 = - X B_{5,X} / 3 \]
Beyond Horndeski

- The Lagrangians

\[ L_2 = A_2 \quad L_3 = A_3 \, K \]
\[ L_4 = A_4 \left( K^2 - K_{ij} K^{ij} \right) + B_4 \, R \]
\[ L_5 = A_5 \left( K^3 - 3K K_{ij} K^{ij} + 2K_{ij} K^{ik} K^{j}_k \right) + B_5 \, K^{ij} \left[ R_{ij} - h_{ij} R/2 \right] \]

with generic coefficients do not lead to any additional d.o.f.

No Ostrogradski instabilities!

Theories with higher time derivatives often contain extra DOF (extra initial conditions), which lead to instabilities.

\[ e.g. \quad L(q, \dot{q}, \ddot{q}) \]

- Explicit Hamiltonian analysis
Beyond Horndeski

• In covariant form, we get extra terms

\[ L_4^\phi = G_4(\phi, X) (^{(4)}R - 2G_4X(\phi, X)(\Box \phi^2 - \phi^{\mu \nu} \phi_{\mu \nu}) \]
\[ + F_4(\phi, X) \epsilon^{\mu \nu \rho \sigma} \epsilon^{\mu' \nu' \rho' \sigma'} \phi_\mu \phi_{\mu'} \phi_{\nu \nu'} \phi_{\rho \rho'} , \]
\[ L_5^\phi = G_5(\phi, X) (^{(4)}G_{\mu \nu} \phi^{\mu \nu} + \frac{1}{3}G_5X(\phi, X)(\Box \phi^3 - 3 \Box \phi \phi_\mu \phi_{\mu \nu} \phi^{\mu \nu} + 2 \phi_\mu \phi^{\mu \sigma} \phi^{\nu}_{\sigma}) \]
\[ + F_5(\phi, X) \epsilon^{\mu \nu \rho \sigma} \epsilon^{\mu' \nu' \rho' \sigma'} \phi_\mu \phi_{\mu'} \phi_{\nu \nu'} \phi_{\rho \rho'} \phi_{\sigma \sigma'} \]

which do not belong to Horndeski class.

• Lagrangians with \( F_4 = 0 \) or with \( F_5 = 0 \) can be connected to Horndeski, via disformal transformations

\[ \tilde{g}_{\mu \nu} = g_{\mu \nu} + \Gamma(\phi, X) \partial_\mu \phi \partial_\nu \phi \]

GLPV 1408
Disformal transformations

• Transformations of the metric

\[ g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2(\phi, X) g_{\mu\nu} + \Gamma(\phi, X) \partial_\mu\phi \partial_\nu\phi \]

• In ADM formalism (with \( t = \phi \)), this gives

\[ \tilde{N}^2 = \Omega^2(t, N) N^2 - \Gamma(t, N), \quad \tilde{N}^i = N^i, \quad \tilde{h}_{ij} = \Omega^2(t, N) h_{ij} \]

• If \( \Omega = \Omega(t) \), the structure of the Lagrangian is preserved

\[ L = A_2 + A_3 K + A_4(K^2 - K_{ij}K^{ij}) + B_4 R \]
\[ + A_5(K^3 - 3K K_{ij}K^{ij} + 2K_{ij}K^{ik}K^{j}_k) + B_5 K^{ij}(R_{ij} - \frac{R}{2} h_{ij}) \]

with new coefficients \( \tilde{A}_i, \tilde{B}_i \)
Disformal transformation & Horndeski

• A generic Lagrangian \( L_4 \) of the form

\[
L = A_2 + A_3 K + A_4 (K^2 - K_{ij}K^{ij}) + B_4 R
\]

can be mapped, via \( \tilde{g}_{\mu\nu} = g_{\mu\nu} + \Gamma_4(\phi, X) \partial_\mu \phi \partial_\nu \phi \)

into a Horndeski Lagrangian \( \tilde{L}_4^H \), i.e. with coefficients \( \tilde{A}_4 \) and \( \tilde{B}_4 \) satisfying

\[
\tilde{A}_4 = -\tilde{B}_4 + 2\tilde{X}\tilde{B}_4\tilde{X}
\]

• The disformal transformation \( \Gamma_4(\phi, X) \) must verify

\[
\Gamma_{4X} = \frac{A_4 + B_4 - 2XB_4X}{X^2 A_4}
\]
Disformal transformation & Horndeski

• The same result holds for $L_5$

$$\Gamma_{5X} = \frac{3A_5 + XB_5X}{3X^2A_5}$$

• Note special disformal transformations with preserve Horndeski’s class.

Note special disformal transformations with preserve Horndeski’s class.  

$\Gamma = \Gamma(\phi)$

Bettoni & Liberati 2013

• Lagrangians with $F_4 = 0$ or with $F_5 = 0$ can be connected to Horndeski, but not those with general coefficients (2 conditions for $\Gamma$).
Modified gravity inside objects

Saito, Yamauchi, Mizuno, Gleyzes & DL 1503.01448
(see also Koyama & Sakstein 1502.06872)

• Partial breaking of Vainshtein mechanism inside matter
  Kobayashi, Watanabe & Yamauchi 2014

• Spherical symmetry & nonrelativistic limit:

\[
\frac{d\Phi}{dr} = G_N \left( \frac{\mathcal{M}}{r^2} - \epsilon \frac{d^2\mathcal{M}}{dr^2} \right), \quad \mathcal{M}(r) = 4\pi \int_0^r r'^2 \rho(r')dr'
\]

• A polytropic equation of state \( P = K \rho^{1+\frac{1}{n}} \) leads to a modified Lane-Emden equation

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left[ \xi^2 \frac{d}{d\xi} \left( \chi - \epsilon \xi^2 \chi^n \right) \right] = -\chi^n \quad \rho \equiv \rho_c [\chi(\xi)]^n, \quad \xi \equiv \frac{r}{r_c}
\]
Modified gravity inside objects

- Spherical profile is modified

\[ M = 4\pi r_c^3 \rho_c \int_0^{\xi_1} \xi^2 [\chi(\xi)]^n d\xi \]
\[ = -4\pi r_c^3 \rho_c \xi_1^2 \left. \frac{d\chi}{d\xi} \right|_{\xi=\xi_1} \quad (n > 1) \]

\[ R = r_c \xi_1 \]

- Universal bound: \( \epsilon < 1/6 \)

Near the center \( (r \ll R) \)

\[ \rho = \rho_c + \frac{1}{2} \rho_2 \frac{r^2}{R^2} + \cdots, \quad P = P_c + \frac{1}{2} P_2 \frac{r^2}{R^2} + \cdots \]

\[ \implies P_2 = -\frac{4\pi G \rho_c^2 R^2}{3} (1 - 6\epsilon) \]
Perturbations in an arbitrary gauge

• Description in an arbitrary slicing?

\[ t \xrightarrow{} t + \pi(t, \vec{x}) \quad \text{(Stueckelberg trick)} \]
Perturbations in the Newtonian gauge

- Perturbed metric

\[ ds^2 = -(1 + 2\Phi)dt^2 + a^2(t) (1 - 2\Psi) \delta_{ij} dx^i dx^j \]

- Modified Einstein’s equations

\[ \delta G^\text{mod}_{\mu\nu} = M^{-2} \delta T^\text{matter}_{\mu\nu} \]

or

\[ \delta G_{\mu\nu} = M^{-2} (\delta T^\text{matter}_{\mu\nu} + \delta T^D_{\mu\nu}) \]

with

\[ \delta \rho_D = \frac{\gamma_1 \gamma_2 + \gamma_3 \alpha_B^2 \tilde{k}^2}{\gamma_1 + \alpha_B^2 \tilde{k}^2} (\delta \rho_D - 3Hq_D) + \frac{\gamma_1 \gamma_4 + \gamma_5 \alpha_B^2 \tilde{k}^2}{\gamma_1 + \alpha_B^2 \tilde{k}^2} Hq_D + \gamma_7 \delta \rho_m \]

\[ \sigma_D = \frac{a^2}{2k^2} \left[ \frac{\gamma_1 \alpha_T + \gamma_8 \alpha_B^2 \tilde{k}^2}{\gamma_1 + \alpha_B^2 \tilde{k}^2} (\delta \rho_D - 3Hq_D) + \frac{\gamma_9 \tilde{k}^2}{\gamma_1 + \alpha_B^2 \tilde{k}^2} Hq_D + \alpha_T \delta \rho_m \right] \]
Testing deviations from GR

• In the quasi-static approximation for sub-horizon scales (time derivatives are neglected), one can express

- effective Newton constant
  \[ -\frac{k^2}{a^2} \Phi \equiv 4\pi G_{\text{eff}}(t, k) \rho_m \Delta_m \]

- slip parameter
  \[ \Psi \equiv \gamma(t, k) \Phi \]

in terms of the coefficients \( \alpha_i \).

• Full system of equations can be implemented in a modified numerical code.
Conclusions

• **Unified treatment** of dark energy and modified gravity models, based on ADM formalism.
  – Easy comparisons between models
  – Identification of degeneracies
  – Observational data can constrain many models simultaneously
  – Explore unchartered territories (e.g. theories beyond Horndeski)

• Very general and efficient way to describe linear perturbations in scalar-tensor theories with **only five time-dependent functions**.