Groups of Coordinate Transformations Between Accelerated Frames Based on the Equivalence Principle
La Thuile, 26 March 2015

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The uniformly accelerated systems related by the equivalence principle to the homogeneous gravitational fields are the issues that are of fundamental importance for general relativity. A discussion of accelerated systems is also essential to obtain a good understanding of special relativity. In both contexts, a description of events and particle motions from the point of view of an observer fixed in an accelerated reference frame is of the main interest.

While the coordinate transformations from inertial to accelerated frames are widely studied in the literature from different physical approaches, there is very little discussion of transformations between accelerated frames.

Conceptually, such transformations should play a central role in the theory of accelerated frames, like as the Lorentz transformations do for inertial frames.

Based on fundamental physical principles, transformations between accelerated frames should possess the group property while for the transformations from inertial to accelerated frames, due to the distinguished role of the former, the group property is not expected.
Main principles

In the present work, transformations between accelerated frames are derived from the two main principles:

- **Group property**
- **Invariance of the interval**

The *interval* is defined by

\[
ds^2 = u(z)^2 dt^2 - \left( \frac{u'(z)}{g} \right)^2 dz^2 - dx^2 - dy^2
\]

where \(g\) is a real constant and \(u(z)\) is an arbitrary real function restricted by requirements of continuity and existence of the small \(g\) limit

\[
u(z) \approx 1 + gz \quad \text{for} \quad gz \to 0
\]

(a system of units in which the light speed \(c = 1\) is used).

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Equivalence principle

- **Equivalence principle**, according to which the metric in the accelerating coordinate system is the same as that of a uniform and constant gravitational field, underlines the form of the interval.

- Although the equivalence principle applies, in general, only at the (infinitesimal) local level, a uniform gravitational field should be globally indistinguishable from a uniformly accelerated frame.

- This equivalence is exact: the freely falling observer concludes that an observer at rest in a uniform gravitational field undergoes the same hyperbolic motion as does the observer with constant proper acceleration in flat spacetime when measured by the inertial observer. The trajectories are identical for all points and all times where the coordinates are valid.
Form of the interval

- The metric

\[ ds^2 = u(z)^2 dt^2 - \left( \frac{u'(z)}{g} \right)^2 dz^2 - dx^2 - dy^2 \]

corresponds to a static homogeneous gravitational field parallel to the z-axis.

- This metric can be derived using the assumptions of homogeneity and isotropy in the field equations of general relativity

\[ R_{\mu\nu} = 0 \]

- For the spatially homogeneous space-times it implies that the space-time is flat

\[ R^\alpha_{\beta \mu \nu} = 0 \]

- The property that the metric is not uniquely defined is not related to the specific problem of the static homogeneous gravitational field and is shared by any solution of the general relativity field equations.

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Transformations between inertial frames (Lorentz transformations) are those leaving invariant the metric of Minkowski space-time

$$ds^2 = (dt')^2 - (dz')^2 - (dx')^2 - (dy')^2$$

Transformations between a uniformly accelerated frame and an inertial frame are coordinate transformations from the metric of a flat space time

$$ds^2 = u(z)^2 dt^2 - \left( \frac{u'(z)}{g} \right)^2 dz^2 - dx^2 - dy^2$$

to a Minkowski space-time ($x', y', z', t'$).

The transformation itself is defined as

$$x' = x, \quad y' = y,$$

$$z' = \frac{u(z)}{g} \cosh gt - \frac{1}{g}, \quad t' = \frac{u(z)}{g} \sinh gt$$

where it has been assumed that one event $O$ is the spacetime origin of both frames and that they are instantaneously at rest with respect to each other at $O$. 

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The transformations between uniformly accelerated frames should be derived as those leaving invariant the metric

\[ ds^2 = u(z)^2 dt^2 - \left( \frac{u'(z)}{g} \right)^2 dz^2 - dx^2 - dy^2 \]

In Hsu and Hsu (1997), Hsu and Kleff (1998), transformations between accelerated frames are derived using a new approach of the limiting 4-dimensional symmetry.

A need in the modification is justified, as follows. The conventional 'gravitational' approach cannot produce transformations that can be smoothly connected to the Lorentz transformations in the limit of zero accelerations since the metric of static homogeneous gravitational field involves only one parameter (acceleration).

Analysis of the present work shows that this argument is unfounded.
Outline of the method

- **One-parameter group of transformations** between two frames $K(T, X, Y, Z, G)$ and $k(t, x, y, z, g)$:

  
  \[ g = h_0(G; a), \]
  \[ t = h_1(T, Z, X, Y, G; a), \]
  \[ z = h_2(T, Z, X, Y, G; a), \]
  \[ x = h_3(T, Z, X, Y, G; a), \]
  \[ y = h_4(T, Z, X, Y, G; a) \]

- **Infinitesimal transformations**

  \[ g \approx G + a \chi(G), \]
  \[ t \approx T + a \tau(T, Z, X, Y, G), \]
  \[ z \approx Z + a \zeta(T, Z, X, Y, G), \]
  \[ x \approx X + a \xi(T, Z, X, Y, G), \]
  \[ y \approx Y + a \eta(T, Z, X, Y, G) \]

- **Condition of infinitesimal invariance**

  \[ ds^2 = dS^2 + a \Delta, \quad ds^2 = dS^2 \Rightarrow \Delta = 0 \Rightarrow \text{group generators} \]

- **Finite transformations** are recovered by exponentiating the infinitesimal generator of the group (solving Lie equations).
- **Change of variables** \((t, z) \rightarrow (f_1, f_2)\):

\[
 f_1 = \frac{u(z)}{g} (\sinh gt + \cosh gt), \quad f_2 = \frac{u(z)}{g} (\sinh gt - \cosh gt)
\]

- **Infinitesimal transformations in new variables**

\[
 g \approx G + a \chi(G),
 f_1 \approx F_1 + a \phi_1(F_1, F_2, X, Y, G), \quad f_2 \approx F_2 + a \phi_2(F_1, F_2, X, Y, G),
 x \approx X + a \xi(F_1, F_2, X, Y, G), \quad y \approx Y + a \eta(F_1, F_2, X, Y, G)
\]

- **Infinitesimal operator** (generator of the group)

\[
 X = \phi_1 \frac{\partial}{\partial f_1} + \phi_2 \frac{\partial}{\partial f_2} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \chi \frac{\partial}{\partial g}
\]

- **Finite transformations**

\[
 h = e^X H; \quad h = \{f_1, f_2, x, y, g\}, \quad H = \{F_1, F_2, X, Y, G\}
\]
One-parameter groups

Finite transformations are found by exponentiating the infinitesimal generator of the group

\[ h = e^{XH} \]

which is equivalent to solving the initial data problem (*Lie equations* with initial conditions):

\[
\frac{df_1(a)}{da} = \phi_1(f_1(a), f_2(a), x(a), y(a), g(a)),
\]

\[
\frac{df_2(a)}{da} = \phi_2(f_1(a), f_2(a), x(a), y(a), g(a)),
\]

\[
\ldots
\]

\[
\frac{dg(a)}{da} = \chi(g(a));
\]

\[ f_1(0) = F_1, \; f_2(0) = F_2, \; x(0) = X, \; y(0) = Y, \; g(0) = G \]
Multi-parameter groups

- If the generator $X$ includes arbitrary parameters, it may be equally considered as defining a multi-parameter group of transformations with the generators $\{X_\alpha\}, \alpha = 1, 2, \ldots, r$.

- For the transformations to form an $r$-parameter Lie group of transformations the corresponding infinitesimal generators must form an $r$-dimensional Lie algebra, with a property of closure with respect to commutation

$$[X_\alpha, X_\beta] = C^\gamma_{\alpha \beta} X_\gamma$$  \hspace{1cm} (1)

- Then an $r$-parameter Lie group of finite transformations is defined as a composition of $r$ one-parameter groups generated by each of base operators $X_\alpha$ via exponentiation or, what is the same, successive solution of the Lie equations

$$h = \prod_{\alpha=1}^{r} e^{\sigma_\alpha X_\alpha} H = e^{\sigma_1 X_1} e^{\sigma_2 X_2} \ldots e^{\sigma_r X_r} H$$
Solutions of determining equations for group generators

\begin{align*}
\phi_1(F_1, F_2, X, Y, G) &= a_0 F_1 + 2\lambda_1 X - 2\lambda_0 Y + \frac{\lambda_4}{G}, \\
\phi_2(F_1, F_2, X, Y, G) &= -a_0 F_2 - 2\lambda_3 X - 2\lambda_2 Y + \frac{\lambda_5}{G}, \\
\xi(F_1, F_2, X, Y, G) &= \lambda_1 F_2 - \lambda_3 F_1 - a_9 Y + \frac{a_7}{G}, \\
\eta(F_1, F_2, X, Y, G) &= -\lambda_0 F_2 - \lambda_2 F_1 + a_9 X + \frac{a_8}{G}, \\
\chi(G) &= G
\end{align*}

where

\begin{align*}
\lambda_0 &= \frac{a_6 - a_5}{2}, \quad \lambda_1 = \frac{a_3 - a_4}{2}, \quad \lambda_2 = -\frac{a_5 + a_6}{2}, \\
\lambda_3 &= -\frac{a_3 + a_4}{2}, \quad \lambda_4 = a_1 - a_2, \quad \lambda_5 = a_1 + a_2.
\end{align*}

and all \( a_i, \ i = 1, 2, \ldots, 9, \) are nondimensional constants.
Two-dimensional (1+1) transformations

Specifications

1. The groups of transformations for which \( x = X \) and \( y = Y \) and transformations of \( z \) and \( t \) do not involve the variables \( x \) and \( y \) are considered.

2. The condition that the same event is a space-time origin of both frames is imposed.

3. The transformations including transformations to an inertial frame as a particular case (allowing nonsingular limit \( g \to 0 \)) are separated.

Conditions 1 and 2 can be imposed on infinitesimals which yields

\[
\phi_1(F_1, F_2, G) = a_0 F_1 - \frac{1 + a_0}{G}, \quad \phi_2(F_1, F_2, G) = -a_0 F_2 + \frac{1 - a_0}{G},
\]

\[
\chi(G) = G
\]

Condition 3 can be used only after identifying finite transformations.
One-parameter groups

- Solution of the Lie equations

\[ g = Ge^a, \quad f_1 = F_1 e^{a_0 a} + \frac{e^{-a} - e^{a_0 a}}{G}, \quad f_2 = F_2 e^{-a_0 a} - \frac{e^{-a} - e^{-a_0 a}}{G}. \]

where \( a \) is a group parameter and \( a_0 \) is a real number.

- It is easily checked that these transformations leave the interval invariant.

- To check whether the transformations allow a nonsingular limit \( g \to 0 \), two first terms of expansions with respect to a small parameter \( \epsilon \)

\[ \epsilon = \frac{g}{G}, \quad a = \ln \epsilon \]

are calculated.

- It is found that the one-parameter group of transformations may include a transformation to an inertial frame only if \( a_0 = 0 \).
Two-dimensional (1+1) transformations forming one-parameter group

\[ t = \frac{1}{g} \text{arccoth} \left( \left( \frac{1}{g} - \frac{1}{G} + \frac{u(Z)}{G} \cosh GT \right) \left( \frac{u(Z)}{G} \sinh GT \right)^{-1} \right) \]

\[ \frac{u(z)}{g} = \sqrt{\left( \frac{1}{g} - \frac{1}{G} \right)^2 + \left( \frac{u(Z)}{G} \right)^2 + 2 \left( \frac{1}{g} - \frac{1}{G} \right) \frac{u(Z)}{G} \cosh GT} \]

In the limit of \( g \to 0 \), expanding the right-hand sides of the expressions up to the order of \( \epsilon = g/G \) yields the transformations to an inertial frame

\[ t' = \frac{u(Z)}{G} \sinh GT, \quad z' = -\frac{1}{G} + \frac{u(Z)}{G} \cosh GT \]
Two-parameter group

\[ f_1 = e^b F_1 + \frac{e^{-a} - e^b}{G}, \quad f_2 = e^{-b} F_2 + \frac{e^{-b} - e^{-a}}{G}, \quad g = Ge^a \]

where \((a, b)\) are the group parameters. The transformations satisfy the group property with the addition law of composition for both group parameters \(a\) and \(b\).

Transformations in the original variables

\[ \mu = \text{arccoth} \frac{G - g \cosh b + gGP \cosh (M + b)}{g (- \sinh b + GP \sinh (M + b))}, \]

\[ p = \sqrt{\frac{1}{g^2} + \frac{1}{G^2} + P^2 - \frac{2 (gP \cosh M + \cosh b - GP \cosh (M + b))}{gG}} \]

\[ p = \frac{u(z)}{g}, \quad \mu = gt; \quad P = \frac{u(Z)}{G}, \quad M = GT \]
Physical meaning of the parameter $b$

The physical meaning of the parameter $b$ becomes clear when one calculates the relative velocity of the space origins of the frames $k$ and $K$ at the initial moment $t = T = 0$ when the origins of both frames coincide. However, to do it the function $u(z)$ is to be specified. The Møller metric and Lass metric were considered.

$$u(z) = 1 + g z \text{ (Møller metric)} \quad u(z) = e^{gz} \text{ Lass metric}$$

For both metrics, calculations give for the velocity $V$ of the origin of $k$ measured by an observer at the origin of $K$ the following

$$V = - \tanh b$$
Transformations in terms of $V$

$$\mu = \text{arccoth} \frac{G + \gamma g (-1 + GP (\cosh M - V \sinh M))}{\gamma g (V + GP (-V \cosh M + \sinh M))},$$

$$p = \sqrt{\frac{1}{g^2} + \frac{1}{G^2} + P^2 - \frac{2\gamma}{gG} + \left( -\frac{2P}{G} + \frac{2P\gamma}{g} \right) \cosh M - \frac{2PV\gamma \sinh M}{g}},$$

where

$$p = \frac{u(z)}{g}, \quad \mu = gt; \quad P = \frac{u(Z)}{G}, \quad M = GT; \quad \gamma = \frac{1}{\sqrt{1 - V^2}}$$

Taking the limit of both small $g$ and $G$ yields the Lorentz transformations

$$z = \frac{Z - VT}{\sqrt{1 - V^2}}, \quad t = \frac{T - VZ}{\sqrt{1 - V^2}}$$
Three-dimensional (2+1) transformations

- After satisfying the requirement that transformations to an inertial frame were included as a particular case, only a two-parameter group remains.
- The transformations belonging to that group correspond to the situation when the direction of the relative velocity of the frame space origins at the initial moment is not along the \( z \)-axis (direction of acceleration).
- In the limit when both frames are inertial, the transformations become the Lorentz boost in arbitrary direction.

Four-dimensional (3+1) transformations

- There remains only a two-parameter group, and the corresponding transformations can be reduced to the three-dimensional case by the coordinate change.
- Thus, the two-parameter group of three-dimensional (2+1) transformations may be considered as a generalization of the Lorentz transformations to accelerated frames in four (3+1) dimensions.
Applications: differential aging between accelerated twins

- At the beginning, both the twin that remains at home (Alice) and the twin that intends to undergo a round-trip journey (Bob) are at rest with respect to the origin of their (in general, accelerated) frame.
- Then Bob leaves Alice acquiring an acceleration with respect to that frame while Alice remains at rest in that frame.
- After the trip, consisting of several segments of accelerated and uniform motion, Bob returns home joining Alice at the origin of her frame with zero relative velocity.
Bob’s trip

- **Segment (1):** Bob starts off and moves with acceleration $G_1 > 0$ (in the positive $z$ direction) during the time interval $T_1$ of his proper time.
- **Segment (1-2):** Bob shuts off his engine and moves inertially during the time interval $T_u$ of his proper time.
- **Segment (2):** Bob starts his engine and decelerates with the acceleration $G_2 < 0$ during the time interval $T_2$.
- **Segment (3):** Bob keeps his engine and accelerates in the negative $z$ direction, with acceleration $G_3 < 0$, during the time interval $T_3$.
- **Segment (3-4):** Bob shuts off his engine and moves inertially during his proper time interval $T_b$.
- **Segment (4):** Bob starts his engine and moves with acceleration $G_4$ coming to rest with respect to Alice’s frame at the space point where Alice is.
Applications: differential aging between accelerated twins

**Parameters**

- $g$ – acceleration of Alice’s frame with respect to an inertial frame
- $G_1, G_2, G_3, G_4$ – Bob’s accelerations at the segments (1), (2), (3), (4) of his trip
- $T_1, T_2, T_3, T_4$ – Bob’s proper time intervals for the segments (1), (2), (3), (4) of accelerated motion
- $T_u, T_b$ – Bob’s proper time intervals for the segments (1-2), (3-4) of inertial motion

**Non-dimensional parameters:**

$$q = \frac{g}{G_1}, \left( \begin{array}{c} T_1 \text{ or } V_1 \text{ or } \gamma = \frac{1}{\sqrt{1-V_1^2}} \end{array} \right), \left( \begin{array}{c} T_3 \text{ or } V_2 \text{ or } \gamma_2 = \frac{1}{\sqrt{1-V_2^2}} \end{array} \right),$$

$$T_4, T_u, T_b, \quad k_2 = - \frac{G_2}{G_1}, \quad k_3 = - \frac{G_3}{G_1}, \quad k = \frac{G_4}{G_1}$$

All the time intervals are made non-dimensional using $G_1$ with the same notation for non-dimensional times:

$G_1 T_1 \rightarrow T_1, \ G_1 T_2 \rightarrow T_2, \ldots$
Twins’ motion with respect to an inertial frame

Figure: Graphs of Bob’s (dashed) and Alice’s (solid) motions in the inertial frame \((z', t')\) for \(q = 0.05\) (left) and \(q = -0.05\) (right): \(\gamma_2 = \gamma = 1.5, T_u = 0.5, T_b = 0.1, k_2 = 1\) and \(k_3 = 2\)

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The differential aging is by definition the difference between the proper times elapsed for the home twin (Alice) and that for the returning twin (Bob) who undergoes a round-trip journey. At the end of his journey Bob joins Alice being at rest in the frame of Alice and they compare readings of their clocks.

So the differential aging $D_a$ is defined by

$$D_a = t - T$$

under the conditions

$$z_{Bob} = 0 \text{ and } V_{Bob} = 0$$

The above conditions are satisfied by adjusting the parameters of Bob’s trip. For $q, \gamma, \gamma_2, T_u, T_b, k_2, k_3$ given, $T_4$ and $k$ are to be determined from the above conditions.
**Differential aging: inertial home twin**

Symmetric formulation

\[ T_b = T_u = 0 \quad V_2 = V_1 \quad \Rightarrow \quad \gamma_2 = \gamma, \quad G_2 = G_3 = -G_1 \quad \Rightarrow \quad k_2 = k_3 = 1 \]

Applying the requirement, that Bob and Alice are reunited at \( z = 0 \) with zero relative velocity, yields

\[ G_4 = G_1 \quad \Rightarrow \quad k = 1 \]

The differential aging is given by

\[ Da = 4 \sqrt{\gamma^2 - 1} - 4 \tanh^{-1} V_1 \]
Differential aging: accelerated home twin

Symmetric formulation -- a counterpart of the 'symmetric' inertial home twin case

\[ T_b = T_u = 0 \quad V_2 = V_1 \Rightarrow \gamma_2 = \gamma, \quad G_2 = G_3 = -G_1 \Rightarrow k_2 = k_3 = 1 \]

Applying the requirement, that Bob and Alice are reunited at \( z = 0 \) with zero relative velocity, yields

\[ k = \frac{1 + q(3 + 4\gamma)}{1 - q(5 + 4\gamma)} \]

The differential aging is given by

\[ Da = \frac{1}{q} \tanh^{-1} \left( \frac{4q \left( 1 + q \left( 1 + 2\gamma \right) \right) \sqrt{\gamma^2 - 1}}{1 + 2q \left( 1 + 2\gamma \right) + q^2 \left( 8\gamma^2 + 4\gamma - 3 \right)} \right) - 3 \tanh^{-1} V_1 \]

\[ -\frac{1}{k} \tanh^{-1} \left( \frac{V_1\gamma \left( 1 + q \left( 4\gamma - 1 \right) \right) \left( 1 + q \left( 4\gamma + 3 \right) \right)}{\gamma + 2q \left( 4\gamma^2 + \gamma - 2 \right) + q^2 \left( -4 - 11\gamma + 8\gamma^2 \left( 1 + 2\gamma \right) \right) \right) \]

In the limit of \( q \to 0 \), it converts into the expression for the inertial home twin case.
Differential aging: accelerated home twin (general formulation)

Figure: Isolines of $Da$ on the plane $(q, \gamma)$ in the region $q_1 < q < q_0$ for $k_2 = 1$, $T_b = 0$, $\gamma_2 = \gamma$; $k_3 = 1$, $T_u = 5$ (left), $k_3 = 4$, $T_u = 10$ (right).
Applications: differential aging between accelerated twins

Differential aging: accelerated home twin (general formulation)

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Accelerated home twin case: parameters of Bob’s motion

- In general, the choice of parameters of Bob’s motion is based on the **distance** to the point of destination and the **time** planned for traveling to that point.
- Distance and time depend on four parameters $G_1$ (or $q$), $V_1$ (or $\gamma$), $T_u$ and $G_2$ (or $k_2$) defining Bob’s motion at the first three segments.
- There are three free parameters $G_3$, $V_2$ (or $\gamma_2$) and $T_b$ defining the motion at the next two segments.
- Thus, there are several freedoms for arranging the trip to a desired object in a desired time interval such that the differential aging were of a desired sign or, in other words, there is a possibility for Bob to choose whether he wishes to find Alice younger or older than himself.

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Conclusions

- The Lie group analysis is applied to determine groups of transformations between accelerated systems. The analysis is based on the equivalence principle according to which the metric in an accelerated frame has the form of static homogeneous gravitational field. Groups of transformations between accelerated frames are found from the condition of invariance of the interval under the transformations.

- For two-dimensional (1+1) transformations, the general result of the analysis is the two-parameter group of transformations which relate accelerated frames with nonzero relative velocity of the space origins at the initial moment. The transformations satisfy the desired property of reducing to the Lorentz transformations when accelerations of both frames vanish, and so may be considered as a proper generalization of the Lorentz boost to accelerated frames.
Conclusions

- In this respect, an argument, that can be found in the literature, that the transformations obtained using the ”gravitational approach” based on the metric of static homogeneous gravitational field cannot be smoothly connected to the Lorentz transformations in the limit of zero accelerations due to the lack of a velocity parameter in the metric, seems unfounded. The analysis of the present paper shows that the velocity parameter does not need to appear in the metric in order to take part in the transformations. It arises as an additional group parameter in the transformations derived through the multi-parameter group analysis.

- The most general transformations between accelerated frames having nonzero relative velocity at the initial moment are given by the three-dimensional (2+1) transformations which represent a generalization of the Lorentz transformations to accelerated frames for the case when the initial relative velocity of the frames is not along the direction of acceleration.

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Figure: A current value of the time dilation factor $\gamma_{T} = (1 - V_{T}^{2})^{-1/2}$ as a function of the inertial time $t'$ for Bob’s (dashed) and Alice’s (solid) motions. The three graphs correspond to the following three points in Fig. 4 (right). Left: $q = 0.1, \gamma = 1.3$ ($Da = -0.21$); Center: $q = 0.1, \gamma = 1.5$ ($Da = 0.27$); Right: $q = 0.02, \gamma = 1.5$ ($Da = 5.1$).