Time-Sliced Perturbation Theory: A new approach to Large Scale Structure

Sergey Sibiryakov

with D. Blas, M. Garny and M.M. Ivanov

arXiv: 1512.05807, 1603.xxxxx

+ work in progress
The beautiful Universe of SDSS
Physics with LSS

• primordial non-gaussianity
  
  interactions in the inflationary sector

• baryon acoustic oscillations = standard ruler in the Universe
  
  dark energy equation of state

• evolution of perturbations
  
  neutrino mass

  properties of dark matter (e.g. fifth force, WDM)
  and dark energy (e.g. clustering)
Challenges of non-linear dynamics

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \]

\[ \nabla_\mu T^{\mu\nu} = 0 \]

Newtonian approximation \( (l \ll 10^4 \text{ Mpc}) \)

+ fluid description \( (l \gg 10 \text{ Mpc}) \)

\[ \frac{\partial \delta \rho}{\partial \tau} + \nabla [(1 + \delta \rho)u] = 0 \]

\[ \frac{\partial u}{\partial \tau} + H(\tau)u + (u \cdot \nabla)u = -\nabla \phi \]

\[ \nabla^2 \phi = \frac{3}{2} \Omega_m(\tau) H^2(\tau) \delta \rho \]
Non-perturbative method: N-body simulations

- advantage: “exact”, goes beyond fluid description
- drawback: too costly -- cannot be used to test many theories beyond the Standard Cosmological Model

Recall that fluid description appears valid up to $k \sim 0.5 \cdot h \cdot \text{Mpc}^{-1}$

use perturbation theory to solve the Euler - Poisson system
Standard perturbation theory (SPT)

\[ \dot{\delta}_\rho(k) - \theta(k) = \int d^3 q \, \alpha(q, k - q) \, \theta(q) \delta_\rho(k - q) \]

\[ \dot{\theta}(k) + \left( \frac{3 \Omega_m}{2 f^2} - 1 \right) \theta(k) - \frac{3 \Omega_m}{2 f^2} \delta_\rho(k) = \int d^3 q \, \beta(q, k - q) \, \theta(q) \theta(k - q) \]
Standard perturbation theory (SPT)

\[
\dot{\delta}_\rho(k) - \theta(k) = \int d^3q \: \alpha(q, k - q) \: \theta(q) \delta_\rho(k - q)
\]

\[
\dot{\theta}(k) + \left( \frac{3\Omega_m}{2f^2} - 1 \right) \theta(k) - \frac{3\Omega_m}{2f^2} \delta_\rho(k) = \int d^3q \: \beta(q, k - q) \: \theta(q) \theta(k - q)
\]

Solve for time evolution iteratively: \( \psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \ldots \)
Standard perturbation theory (SPT)

\[
\dot{\delta}_\rho(k) - \theta(k) = \int d^3 q \, \alpha(q, k - q) \, \theta(q) \delta_\rho(k - q)
\]

\[
\dot{\theta}(k) + \left( \frac{3 \Omega_m}{2 f^2} - 1 \right) \theta(k) - \frac{3 \Omega_m}{2 f^2} \delta_\rho(k) = \int d^3 q \, \beta(q, k - q) \, \theta(q) \theta(k - q)
\]

Solve for time evolution iteratively:

\[\psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \ldots\]

\[\psi^{(1)} = \psi_0\]

evolution from \(\tau\) to \(\tau_0\)
Standard perturbation theory (SPT)

\[ \dot{\delta}_\rho(k) - \theta(k) = \int d^3q \, \alpha(q, k - q) \, \theta(q) \delta_\rho(k - q) \]

\[ \dot{\theta}(k) + \left( \frac{3\Omega_m}{2f^2} - 1 \right) \theta(k) - \frac{3\Omega_m}{2f^2} \delta_\rho(k) = \int d^3q \, \beta(q, k - q) \, \theta(q) \theta(k - q) \]

Solve for time evolution iteratively:

\[ \psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \ldots \]

\[ \psi^{(1)} = \psi_0 \]

\[ \psi^{(2)} = \psi_0 \]

Evolution from \( \tau \) to \( \tau_0 \)
Standard perturbation theory (SPT)

\[ \dot{\delta}_\rho(k) - \theta(k) = \int d^3q \, \alpha(q, k-q) \, \theta(q) \delta_\rho(k-q) \]

\[ \dot{\theta}(k) + \left( \frac{3\Omega_m}{2f^2} - 1 \right) \theta(k) - \frac{3\Omega_m}{2f^2} \delta_\rho(k) = \int d^3q \, \beta(q, k-q) \, \theta(q) \theta(k-q) \]

Solve for time evolution iteratively:
\[ \psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)} + \ldots \]

\[ \psi^{(1)} = \psi_0 \]

\[ \psi^{(2)} = \psi_0 \]

\[ \psi^{(3)} = \psi_0 \]
Average over the ensemble of initial conditions:

\[ \langle \psi(k_1, \tau)\psi(k_2, \tau) \rangle = \langle \psi^{(1)} \psi^{(1)} \rangle + \langle \psi^{(2)} \psi^{(2)} \rangle + 2\langle \psi^{(1)} \psi^{(3)} \rangle + \ldots = \]

\[ \begin{array}{c}
\text{initial power spectrum } P(k, \tau_0)
\end{array} \]
Problems of SPT

“Ultraviolet” Loop integrals run over all momenta including short modes where the fluid description is not applicable.
Problems of SPT

“Ultraviolet” Loop integrals run over all momenta including short modes where the fluid description is not applicable.

1) introduce a UV cutoff $\Lambda$
Problems of SPT

“Ultraviolet” Loop integrals run over all momenta including short modes where the fluid description is not applicable.

1) introduce a UV cutoff $\Lambda$

2) renormalize the interaction vertices to ensure that the physical observables are $\Lambda$-independent
Problems of SPT

“Ultraviolet” Loop integrals run over all momenta including short modes where the fluid description is not applicable.

1) introduce a UV cutoff $\Lambda$

2) renormalize the interaction vertices to ensure that the physical observables are $\Lambda$-independent

3) add counterterms into the equations of motion to account for deviations from fluid description
Problems of SPT

“Ultraviolet” Loop integrals run over all momenta including short modes where the fluid description is not applicable.

1) introduce a UV cutoff $\Lambda$

2) renormalize the interaction vertices to ensure that the physical observables are $\Lambda$-independent

3) add counterterms into the equations of motion to account for deviations from fluid description

EFT of LSS  

Carrasco, Hertzberg, Senatore (2012)
Problems of SPT

“Ultraviolet” Loop integrals run over all momenta including short modes where the fluid description is not applicable.

1) introduce a UV cutoff $\Lambda$

2) renormalize the interaction vertices to ensure that the physical observables are $\Lambda$-independent

3) add counterterms into the equations of motion to account for deviations from fluid description

Complications:
- counterterms have non-local time-dependence

EFT of LSS Carrasco, Hertzberg, Senatore (2012)

Abolhasani, Mirbabayi, Pajer (2015)
Problems of SPT

“Ultraviolet” Loop integrals run over all momenta including short modes where the fluid description is not applicable.

1) introduce a UV cutoff $\Lambda$

2) renormalize the interaction vertices to ensure that the physical observables are $\Lambda$-independent

3) add counterterms into the equations of motion to account for deviations from fluid description

EFT of LSS \textit{Carrasco, Hertzberg, Senatore (2012)}

Complications:

• counterterms have \textit{non-local time-dependence} \textit{Abolhasani, Mirbabayi, Pajer (2015)}

• treatment of \textit{stochastic} terms is unclear
Problems of SPT

“Infrared” Kernels $\alpha, \beta$ in the e.o.m.’s behave as $1/q$

Individual loop diagrams diverge at small momenta

When summed, the divergences cancel in equal-time correlators
Problems of SPT

“Infrared” Kernels $\alpha, \beta$ in the e.o.m.’s behave as $1/q$

individual loop diagrams diverge at small momenta

When summed, the divergences cancel in equal-time correlators

overdensity is moved by an almost homogeneous flow

accumulation of the effect with time
Problems of SPT

“Infrared” Kernels $\alpha, \beta$ in the e.o.m.’s behave as $1/q$

individual loop diagrams diverge at small momenta

When summed, the divergences cancel in equal-time correlators

overdensity is moved by an almost homogeneous flow

accumulation of the effect with time

two overdensities will move (almost) identically

cancellation in equal-time correlators
TSPT: time-sliced perturbation theory

Main ideas: Focus on equal-time correlators
Evolve the whole statistical ensemble in time
TSPT: time-sliced perturbation theory

Main ideas: Focus on equal-time correlators
Evolve the whole statistical ensemble in time

Example: Consider a single variable with random initial conditions

\[ \dot{\psi} = \Omega \psi + \sum_{m=2} A_m \frac{m!}{m!} \psi^m \]
\[ \psi(\tau; \psi_0) \]
TSPT: time-sliced perturbation theory

Main ideas: Focus on equal-time correlators
Evolve the whole statistical ensemble in time

Example: Consider a single variable with random initial conditions

\[
\dot{\psi} = \Omega \psi + \sum_{m=2}^{\infty} \frac{A_m}{m!} \psi^m \quad \rightarrow \quad \psi(\tau; \psi_0)
\]

SPT: \[
\int d\psi_0 \ e^{-\Gamma_0[\psi_0]} \psi(\tau; \psi_0)^2 \quad \Gamma_0[\psi_0] = \frac{\psi_0^2}{2P}
\]
**TSPT: time-sliced perturbation theory**

**Main ideas:** Focus on equal-time correlators

Evolve the whole statistical ensemble in time

**Example:** Consider a single variable with random initial conditions

\[ \dot{\psi} = \Omega \psi + \sum_{m=2} \frac{A_m}{m!} \psi^m \Rightarrow \psi(\tau; \psi_0) \]

**SPT:**
\[ \int d\psi_0 \ e^{-\Gamma_0[\psi_0]} \psi(\tau; \psi_0)^2 \]

\[ \Gamma_0[\psi_0] = \frac{\psi_0^2}{2P} \]

**TSPT:**
\[ \int d\psi \ e^{-\Gamma[\psi;\tau]} \psi^2 \]

\[ \Gamma[\psi; \tau] = \sum_n \frac{\Gamma_n(\tau)}{n!} \psi^n \]
Two integrals must coincide

\[ \frac{d}{d\tau} \left( d\psi e^{-\Gamma[\psi;\tau]} \right) = 0 \]

\[ \dot{\Gamma}_n = -n\Omega\Gamma_n - \sum_{m=2}^{n} C_n^m A_m \Gamma_{n-m+1} + A_{n+1} \]

contains only \( \Gamma_{n'} \) with \( n' < n \)
Two integrals must coincide

equation for the “vertices”

\[ \frac{d}{d\tau} \left( d\psi e^{-\Gamma[\psi;\tau]} \right) = 0 \]

\[ \dot{\Gamma}_n = -n\Omega\Gamma_n - \sum_{m=2}^{n} C_n^m A_m \Gamma_{n-m+1} + A_{n+1} \]

contains only \( \Gamma_{n'} \) with \( n' < n \)

The same logic for fields in space with the substitution: integral \( \rightarrow \) path integral
Generating functional for cosmological correlators

\[ Z[J, J_\delta; \tau] = \int [\mathcal{D} \theta] \exp \left\{ - \Gamma[\theta; \tau] + \int \theta J + \int \delta_\rho[\theta; \tau] J_\delta \right\} \]

\[ \Gamma = \frac{1}{2} \int \frac{\theta^2}{\hat{P}(k)} + \sum_{n=3}^{\infty} \frac{1}{n!} \int \Gamma_n(\tau) \theta^n \]

\[ \delta_\rho = \sum_{n=1}^{\infty} \frac{1}{n!} \int K_n(\tau) \theta^n \]

TSPT - 3d Euclidean QFT vocabulary:

- \( \Gamma \) --- 1PI effective action
- \( \delta_\rho \) --- composite source
- \( \tau \) --- external parameter
Advantages

• For gaussian initial conditions the time dependence factorize

\[ \Gamma = \frac{1}{g^2(\tau)} \tilde{\Gamma} \]

effective coupling constant

**NB.** For primordial NG

\[ \Gamma = \frac{1}{g^2} \tilde{\Gamma} + \frac{1}{g^3} \hat{\Gamma} \]

\[ \sim f_{NL} g_0 \]
• Simplified diagrammatic technique

\[ \frac{k}{k_2} = g^2 \bar{\mathcal{P}}(k) \]

\[ \frac{k_1}{k_2} = \frac{1}{g^2 \bar{\Gamma}_3(k_1, k_2)} \]

\[ k_1 \cross k_3 = \frac{1}{g^2 \bar{\Gamma}_4(k_1, k_2, k_3)} \]
• Simplified diagrammatic technique

\[ \frac{k}{k_2} = g^2 \bar{P}(k) \]

\[ \frac{k_1}{k_2} = \frac{1}{g^2 \bar{\Gamma}_3(k_1, k_2)} \]

\[ \frac{k_1}{k_2} = \frac{1}{g^2 \bar{\Gamma}_4(k_1, k_2, k_3)} \]

\[ \langle \theta \theta \theta \theta \rangle = \]

\[ + \]

\[ + \]
• Simplified diagrammatic technique

\[ k = g^2 \bar{P}(k) \]

\[ \frac{k_1}{k_2} = \frac{1}{g^2 \bar{\Gamma}_3(k_1, k_2)} \]

\[ \frac{k_1}{k_3} = \frac{1}{g^2 \bar{\Gamma}_4(k_1, k_2, k_3)} \]

\[ \langle \theta \theta \theta \theta \rangle = \quad + \quad \]

**NB.** Different from truncation of the Bogolyubov hierarchy
• Simplified diagrammatic technique

\[ k = g^2 \bar{P}(k) \]

\[ \frac{k_1}{k_2} = \frac{1}{g^2} \bar{\Gamma}_3(k_1, k_2) \]

\[ k_1 \times k_2 = \frac{1}{g^2} \bar{\Gamma}_4(k_1, k_2, k_3) \]

\[ \langle \theta \theta \theta \theta \rangle = \]  

\[ = K_n(k_1, k_2, \ldots, k_n) \]

NB. Different from truncation of the Bogolyubov hierarchy
IR safety

All $\Gamma_n, K_n$ are finite for soft momenta

$$\lim_{\epsilon \to 0} \Gamma_n(k_1, \ldots, k_l, \epsilon q_1, \ldots, \epsilon q_{n-l}) < \infty$$

no IR divergences in the individual loop diagrams

NB. Can be related to the equivalence principle / Galilean invariance of $\Gamma$ through Ward identities
Baryon acoustic oscillations

\[ \langle \delta T(x) \delta T(y) \rangle \]
Baryon acoustic oscillations

\langle \delta T(x) \delta T(y) \rangle

Credit: ESA and the Planck collaboration
Baryon acoustic oscillations

FIG. 4: Measured power spectra for the full LRG and main galaxy samples. Errors are uncorrelated and full window functions are shown in Figure 5. The solid curves correspond to the linear theory $\Lambda$CDM fits to WMAP3 alone from Table 5 of [7], normalized to galaxy bias $b_0$ and $b_1$ relative to the $z=0$ matter power. The dashed curves include the nonlinear correction of [29] for $A=1.4$, with $Q_{nl}=30$ for the LRG and $Q_{nl}=46$ for the main galaxies; see equation (4). The extent of the nonlinear corrections is clearly visible for $k \sim 0.09 h/\text{Mpc}$ (vertical line).

Our Fourier convention is such that the dimensionless power $\Delta^2$ of [77] is given by $\Delta^2(k) = \frac{4\pi}{k^3} P(k)$.

Before using these measurements to constrain cosmological models, one faces important issues regarding their interpretation, related to evolution, nonlinearities and systematics.

B. Clustering evolution

The standard theoretical expectation is for matter clustering to grow over time and for bias (the relative clustering of galaxies and matter) to decrease over time [78–80] for a given class of galaxies. Bias is also...
Baryon acoustic oscillations

FIG. 4: Measured power spectra for the full LRG and main galaxy samples. Errors are uncorrelated and full window functions are shown in Figure 5. The solid curves correspond to the linear theory ΛCDM fits to WMAP3 alone from Table 5 of [7], normalized to galaxy bias.

Our Fourier convention is such that the dimensionless power ∆^2 of [77] is given by ∆^2(k) = 4π(k/2π)^3 P(k).

Before using these measurements to constrain cosmological models, one faces important issues regarding their interpretation, related to evolution, nonlinearities and systematics.

B. Clustering evolution

The standard theoretical expectation is for matter clustering to grow over time and for bias (the relative clustering of galaxies and matter) to decrease over time [78–80] for a given class of galaxies. Bias is also...
IR sensitivity of the BAO peak

\[ \xi(x, z=0) \]

- TSPT NLO
- TSPT LO
- SPT 1-loop
- linear
- N-body

\[ x \text{ [Mpc/h]} \]

- 0.0005
- 0.0000
- 0.0005
- 0.0010
- 0.0015
- 0.0020
- 0.0025

- 0.0015
- 0.0010
- 0.0005
- 0.0000
- 0.0005
- 0.0010
- 0.0015
- 0.0020
- 0.0025

- 140
- 130
- 120
- 110
- 100
- 90
- 80
IR resummation

In TSPT large IR contributions can be systematically resummed

**Step 1:** smooth + wiggly decomposition

\[ P(k) = P_s(k) + P_w(k) \quad \rightarrow \quad \Gamma(k) = \Gamma_s(k) + \Gamma_w(k) \]
**IR resummation**

In TSPT large IR contributions can be systematically resummed

**Step I:** smooth + wiggly decomposition

\[ P(k) = P_s(k) + P_w(k) \]

**Step II:** identification of leading diagrams

\[ P_{\text{dressed}}^{w} = \]

\[
\begin{align*}
\text{linear} &+ \quad \text{bubble} \quad \text{bubble} \quad \text{bubble} \quad \text{bubble} \quad \ldots
\end{align*}
\]
**Step III: add the smooth part**

$$P(k) = P_s(k) + e^{-k^2 \Sigma_L^2} P_w(k)$$

$$\Sigma_L^2 = \frac{4\pi}{3} \int_0^{k_L} dq \, P_s(q) \left( 1 - j_0(q r_s) + 2 j_2(q r_s) \right)$$

**BAO wavelength**

**Baldauf, Mirbabayi, Simonovic, Zaldarriaga (2015)**

**Further developments:**

- take into account NLO IR corrections
- IR resummation of higher-point correlation functions
In the following we show results for the correlation function, because it exhibits a clear separation between the BAO peak and the small-distance part of the correlations, and allows to visualize the effects on the BAO feature in a transparent way. In Fig. 5, we show the leading-order IR resummed result for three different choices of $k_L$ (blue lines). The damping of the BAO oscillations described by $\Delta L$ gives already a relatively good description of the $N$-body result (red line), especially when compared to the linear prediction (dashed lines). Nevertheless, there are some differences, and the dependence on $k_L$ is not negligible.

$k_L$ --- the IR separation scale
Comparison with N-body: NLO

1-loop IR resummed, $z=0$

1-loop IR resummed, $z=0.375$

$k_L$ --- the IR separation scale
UV renormalization in TSPT
UV renormalization in TSPT
UV renormalization in TSPT

Introduce a hard cutoff:

\[ \hat{P}(k) \leftrightarrow \hat{P}^\Lambda(k) = \begin{cases} \hat{P}(k), & k < \Lambda \\ 0, & k > \Lambda \end{cases} \]

\[ \Gamma_n \leftrightarrow \Gamma_n^\Lambda \]
UV renormalization in TSPT

Introduce a hard cutoff:

\[ \hat{P}(k) \mapsto \hat{P}^\Lambda(k) = \begin{cases} 
\hat{P}(k), & k < \Lambda \\
0, & k > \Lambda 
\end{cases} \]

\[ \Gamma_n \mapsto \Gamma_n^\Lambda \]

Wilsonian RG:

\[ \frac{d\Gamma_n^\Lambda}{d\Lambda} = \mathcal{F}_n[\hat{P}^\Lambda, \Gamma^\Lambda] \]

Boundary conditions = counterterms encapsulating the effects of short modes
Comparison with EFT of LSS

+ the cutoff dependence is treated systematically

+ counterterms are manifestly local in time

+ stochastic contributions are at the same footing as the dynamical ones
Comparison with EFT of LSS

+ the cutoff dependence is treated systematically

+ counterterms are manifestly local in time

+ stochastic contributions are at the same footing as the dynamical ones

- spatial locality is not manifest

exploit symmetries in momentum space

+ factorization ????
Summary and Outlook

- time-sliced perturbation theory (TSPT) is a promising approach to LSS in the mildly non-linear regime $20 \, \text{Mpc} < l < 100 \, \text{Mpc}$
- free from IR divergences
- diagrammatic resummation of IR-enhanced contributions into the BAO feature
- UV renormalization à la Wilsonian RG
time-sliced perturbation theory (TSPT) is a promising approach to LSS in the mildly non-linear regime $20 \text{ Mpc} < l < 100 \text{ Mpc}$

free from IR divergences

diagrammatic resummation of IR-enhanced contributions into the BAO feature

UV renormalization à la Wilsonian RG

classification of UV counterterms

redshift space

biases

inclusion of baryons