Pion transition form factor: interplay of soft and hard limits.

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1. Extended Nonlocal NJL model.
2. Quark Propagator and Gap Equation
4. Conserved Currents in nonlocal model.
5. $f_0$ and Goldberger-Treiman relation.
6. Axial Anomaly.
7. Asymptotics of $f_{\gamma^\star \gamma^\star}$ form factor.
8. Distribution Amplitude of pion.
9. $f_{\gamma^\star \gamma^\star}$ form factor.
Motivations

1. Pion Charge Form Factor

\[ Q^2 F(Q^2) \]

\[ \pi^+ \pi^- \to \pi^+ \pi^- \sim N Q^4 \]

\[ \sim 1/Q^2 \]

\[ \sim 10 GeV^2 \]

\[ \sim 100 GeV^2 \]

2. \( \gamma \gamma \to \pi \pi \) at large invariant masses.

3. Pion transition form factors \( \pi \to \gamma^* \gamma^* \)
Form Factor

\[ \begin{align*}
M(y^*(q_1, \varepsilon_1)y^*(q_2, \varepsilon_2) \rightarrow \pi(\mathbf{p})) &= e^2 \varepsilon_{\mu \nu \alpha \beta} \varepsilon_1^\mu \varepsilon_2^\nu q_1^\alpha q_2^\beta T(q_1^2, q_2^2, \mathbf{p}^2) \\
\text{In the chiral limit} & \quad \rho^2 = m_{\pi}^2 = 0 \\
T(0, 0, 0) &= \frac{1}{4 \pi^2} \text{Axial Anomaly}
\end{align*} \]

In the symmetric asymptotic limit:

\[ T(\frac{1}{2} q_1^2, \frac{1}{2} q_2^2, m_{\pi}^2) = \begin{cases} 
\frac{4}{3} \frac{f_{\pi}}{Q^2} & \text{pQCD} \\
\frac{M_q^2}{4 \pi^2 f_{\pi}} \ln \left( \frac{Q^2}{M_q^2} \right) & \text{quark triangle loop}
\end{cases} \]

In the asymmetric asymptotic limit:

\[ T(q_1^2, 0, 0) = \frac{1}{-q_1^2 - \infty} \int \frac{d\mathbf{p}}{Q^2} \left\{ \frac{M_q^2}{4 \pi^2 f_{\pi}} \ln \left( \frac{Q^2}{M_q^2} \right) \right\} \]

\[ J = \frac{4}{3} \int \frac{dx}{x} \frac{\mathcal{L}_{\pi} \rho(x)}{x} = \int \frac{2}{10/3} \text{in asymptotic pQCD} \]
Extended Nonlocal NJL model: $SU(2)$

The action:

\[
S = \int d^4x \bar{\psi}(i\slashed{D} - m) \psi(x) + \\
\sum_{i} \frac{1}{2} G_i \int d^4z \int d^4x_1 K(x_i, x_2, x_3, x_4) \{ \bar{\psi}(2, x_i, 2) \Gamma_i \psi(2, x_2, x_3, x_4) \cdot \bar{\psi}(2, x_i, 2) \Gamma_i \psi(2, x_2, x_3, x_4)\}.
\]

$SU(2) \times SU(2)_A \times U(1)_V$ symmetry requires:

\[
(\Gamma_1 \times \Gamma_1) G_i = \begin{cases} 
G_1 (1 + \gamma^5) & \text{if } i = 1, 2, 4 \\
G_2 (\gamma^0 + i \gamma^3) & \text{if } i = 1, 2, 4 \\
G_3 (\gamma^3) & \text{if } i = 3 \\
G_4 (\gamma^3 \times \gamma^5) & \text{if } i = 3
\end{cases}
\]

Take kernel $K(x_i)$ in separable form

\[
\tilde{K}(x_i, x_2, x_3, x_4) = \tilde{f}(x_i) \tilde{f}(x_2) \tilde{f}(x_3) \tilde{f}(x_4)
\]

as it is appeared in the Instanton Model.

The action $S$ is given in Gauge Invariant form with definitions

\[
D_{\mu} = \partial_{\mu} - i V_{\mu}(x) - i A_{\mu}(x) \gamma_5
\]

\[
\tilde{\psi}(x,y) = \frac{\tau^3}{2} \psi(x,y)
\]

\[
\tilde{\psi}(x,y) = \text{Pexp} \left[ -i \int d^4z \left( V_{\mu}(z) + A_{\mu}(z) \gamma_5 \right) \right] \psi(z)
\]
Quark Propagator, Gap Equations, Nonlocal Condensate, Dynamical Quark Mass.

The dynamical quark mass is defined from the dressed quark propagator

\[ S^{-1}(k) = \hat{k} - M(k) \quad \text{as} \quad k \to \infty \]

The Schwinger-Dyson equation treated in the leading order of \( \frac{1}{N_c} \) expansion / ladder approximation can be written as

\[ M(p) = m_c + i G_1 f^2(p) \int \frac{dk}{(2\pi)^3} T_0 \{ S(k) \} f^2(k) \]

\[ S = S_0 + G_1 S \]

* This form of G.E. corresponds to separable kernel \( (K(k)) \)

The solution of G.E. has a form

\[ M(p) = m_c + (M_g - m_c) f^2(p) \]

with constant \( M_g \) ("constituent quark mass") defined by G.E.

* Under the Isotensor Model

\[ \frac{M_g^2}{G_1} = N_c \approx 0.0016 \text{ GeV}^4 \]
Conserved Currents in Nonlocal Models

In the presence of nonlocal interaction the conserved currents and corresponding vertices include local and nonlocal pieces.

\[ j^\mu(x) = j^\mu_{\text{loc}}(x) + j^{\mu}_{\text{nonloc}}(x) \]

\[ j^\mu_{\text{loc}}(x) = \bar{\Psi}(x) \gamma^\mu \frac{ig^a}{2} \gamma^5 \Psi(x), \quad j^\mu_{\text{nonloc}}(x) = i\bar{\Psi}(x) \gamma^\mu \gamma^5 \frac{ig^a}{2} \gamma^5 \Psi(x) \]

The nonlocal part may be classified as follows:

\[ j^{\mu}_{\text{nonloc}}(q) = \sum_i c_i \int_0^1 \frac{dk_i}{(2\pi)^2} \bar{\Psi}_{k_1} \gamma^\mu \Psi_{k_2} \bar{\Psi}_{k_3} \gamma^\mu \Psi_{k_4} \delta(k_1 + k_2 + q - k_3 - k_4) \]

\[ \cdot \int d\lambda f(k_1) f(k_2) \frac{2}{\partial(k_1 + k_2)} f(k_3 + \lambda q) f(k_3 - \lambda q) \]

**IS-V** \( \gamma_\mu \): \( \Gamma \times \mathbb{R} \) as in original 4-quark Interaction

**IV-V** \( j^{\mu}_{\text{nonloc}} \):

\( G_1 (\gamma^a \gamma^5 \gamma^i + i \gamma^i \gamma^5 \gamma^a) \), \( G_2 (\gamma^i \gamma^j \gamma^5 \gamma^k \gamma^5 \gamma^l) \), \( G_3 (\gamma^i \gamma^j \gamma^k \gamma^5 + i \gamma^k \gamma^5 \gamma^i) \), \( G_4 (\gamma^i \gamma^j \gamma^k \gamma^5 \gamma^a \gamma^b) \), \( G_5 (\gamma^i \gamma^j \gamma^k \gamma^5 \gamma^a \gamma^b \gamma^c) \)

**IV-A** \( j^{\mu}_{\text{nonloc}} \):

\( G_1 \varepsilon^{abc} (\gamma^c \gamma^5 \gamma^a) \), \( G_2 (\gamma^i \gamma^j \gamma^k \gamma^5 \gamma^a \gamma^b \gamma^c) \), \( G_3 (\gamma^i \gamma^j \gamma^k \gamma^5 \gamma^a \gamma^b \gamma^c) \), \( G_4 (\gamma^i \gamma^j \gamma^k \gamma^5 \gamma^a \gamma^b \gamma^c) \), \( G_5 \varepsilon^{abc} (\gamma^i \gamma^j \gamma^k \gamma^5 \gamma^a \gamma^b \gamma^c) \), \( G_6 \varepsilon^{abc} (\gamma^i \gamma^j \gamma^k \gamma^5 \gamma^a \gamma^b \gamma^c) \)

**IV-V**
\[ \Gamma_{\mu}(k,q) = \Gamma_{\mu} - 2Q \int \frac{d^4k'}{(2\pi)^4} \frac{f(k_+ + q') f(k_+)}{2(k_+ q')} \Gamma_{\mu}(k_+) \Gamma_{\mu}(k_+) \]

By using Gap Equation \( G_4 \propto \frac{3}{e} \) reduced to 1.

\[ f(k,q) = \chi_{-} - 2Q \int \frac{d^4k'}{(2\pi)^4} (k_+ + q') M'(k_+ + q') \]

satisfies the Ward-Takahashi identity (WTI)

\[ q_\mu \Gamma_{\mu}(k,q) = S_{-}(k_+) - S_{-}(k_-) \]

\[ \Delta \Gamma_{\mu}(k,q) \equiv \Gamma_{\mu}(k,q) - \frac{1}{2} \Gamma_{\mu}(k,q) \Gamma_{\mu}(k,q) \]

\[ Q = \frac{1}{2} (\tau^3 + \frac{3}{3}) \]

\[ B(q^2) = \frac{1}{2} \left[ \tau^3 B_2(q^2) + \frac{1}{3} B_3(q^2) \right] \]

\[ B(q^2 = 0) = 0 \quad \text{doesn't violate WTI.} \]

\[ B_3(q^2) = \frac{1}{1 - G_{\mu} \Gamma_{\mu}(q^2)} \]

\[ i G_{\mu} N_c N_f \Gamma_{\mu}(q^2) \]

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Meson Propagators. \textit{Ladder Approx.}

The meson masses and vertex functions are found from Bethe-Salpeter equation.

For separable kernel $K$ the $q q$ scattering matrix $T$

$$T(k_1, k_2, k_3, k_4) = \prod \delta(k_{14}) \delta(k_1 - k_3 - k_4) \hat{T}(q)$$

where $q = k_1 - k_3 = k_4 - k_2$. Then BSE

$$\hat{T}(q) = G + GJ(q)\hat{T}(q)$$

has a solution

$$\hat{T}(q) = \frac{G}{1 - GJ(q)}$$

where

$$J_g(q^2) = i \int \frac{d^4 k}{(2\pi)^4} f(k_1) f(k_2) Tr \left[ \Gamma^2 S(k_1) \Gamma^2 S(k_4) \right]$$

BS: $1 \times 1$
PP: $i \gamma_5 \chi \gamma_5$, AP: $-\frac{i \gamma^\nu}{IP} \gamma_5 \chi \gamma_5$
VV$^T$: $T_{\mu\nu} = \gamma_5 \gamma_5 \gamma_5$
VV$^L$: $- \frac{i \gamma^\nu}{IP} \chi \gamma_5$
AA$: $- \frac{i \gamma^\nu}{IP} \gamma_5 \chi \gamma_5$

$$T_{\mu\nu} = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$$

Pole Condition

$$\det (1 - GJ(q)) \frac{q^2}{m^2} = 0$$

defines the meson state spectrum.
Meson Propagators

Near the pole, $\tilde{\tau}$ may be represented as

$$\vec{V}(p) \cdot V(p) \over m^2 - p^2$$

$$V_\pi^\alpha(p) = g_\pi i\gamma^\alpha \gamma^5 - g_\pi \frac{1}{2m} i\gamma^\alpha \gamma^5$$

$$V_\rho^\alpha(p) = g_\rho \gamma^\alpha$$

$$V_\omega^\alpha(p) = g_\omega \gamma^\alpha$$

$$V_{\pi^*}^\alpha(p) = g_{\pi^*} i\gamma^\alpha - \frac{m^2}{2m} i\gamma^\alpha$$

$$V_{\rho^*}^\alpha(p) = g_{\rho^*} \gamma^\alpha$$

$$V_{\omega^*}^\alpha(p) = g_{\omega^*} \gamma^\alpha$$

The quark-meson couplings $g$ are defined from the Pole Condition (S-meson spin)

$$g_i^2 = (-1)^i \frac{dJ_i^\gamma}{dq^2} \bigg|_{q^2 = m_i^2}$$

and for the mixed $\pi\rho$ state:

$$g_{\pi\rho} = \frac{-D_{\pi\rho}^i(m_i^2)}{G_i(1 - G_2 J_{AA}^i)}$$

$$g_{\pi^*\rho^*} = \frac{D_{\pi^*\rho^*}^i(m_i^2)}{G_i G_2 J_{AP}^i(m_i^2)}$$

$$D(q^2) = \text{det} (1 - G) = (1 - G_4 J_{pp}(q^2))(1 - G_2 J_{AA}^i(q^2)) - G_4 G_2 J_{AP}^i(q^2)$$

$$D_{ji}(m_i^2) = 0$$

$$g_{\pi\rho} \sim 0(1)$$

$$g_{\pi^*\rho^*} / m_i \sim 0(1)$$

Pole Conditions (1) and (2) are important for the prove of Goldberger-Treiman Relation and the Axial Anomaly.
The pion decay constant defined as

\[-i f_\pi \, P_m = \langle 0 | j_5^\mu (0) | \pi^- (p) \rangle \]

is sum of contributions:

\[ V_3^a = (g_\pi \gamma_5 - g_\delta \frac{i \gamma_5}{2m} \gamma_5)^a \cdot f + f^- \]

**G_{\pi} sector**

\[ f_\pi = \frac{i}{2p^2} \frac{g_{\pi}}{2m} \int \frac{d^4k}{(2\pi)^4} \, f(k^+) f(k^-) \, Tr \left[ i \gamma_\mu \gamma_5 \gamma_\mu' \gamma_5' \, S(k^+) S(k^-) \right] \]

\[ = \frac{g_{\pi}}{M} \cdot \frac{N_c}{8 \pi^2} \int d\omega \, \rho(\omega) \, \frac{2(2\omega^2 - \omega M M')}{(\omega + m)^2} \]

\[ f_\delta = \frac{i}{2p^2} \frac{g_{\delta} G_2}{2m} \int \frac{d^4k}{(2\pi)^4} \, Tr \left[ S(k^+) S(k^-) \right] \frac{4\pi^2}{(2\pi)^4} \, f(k^+) f(k^-) \, Tr \left[ \gamma_5 \gamma_\mu \gamma_5' \gamma_\mu' \, S(k^+) S(k^-) \right] \cdot \left[ f^0 \, f^2 (k^+) + f^2 (k^-) - f^0 (k^+) \, f^0 (k^-) \right] \]

\[ = \frac{g_{\delta}}{M} \cdot \frac{N_c}{16 \pi^2} \int d\omega \, \frac{\omega^2 M^2 - 2MM' - \omega MM'}{\omega + M} \]

\[ f_\pi = f_\pi^{(a)} + f_\pi^{(b)} = \frac{g_{\pi}}{M} \sqrt{M^2 - 2MM' + M^2 M'^2} \]

Goldberger-Treiman Relation

\[ f_\delta = \frac{N_c}{4 \pi^2} \int d\omega \, \rho(\omega) \, \frac{M^2 - \omega M M' + \omega M'^2}{(\omega + M)^2} \]
\[ \Gamma^0 \to \gamma\gamma \quad (\text{Axial Anomaly}) \]

\[ \text{A} \quad \pi_2 \text{ sector} \]

\[ \Gamma_1 = 4iE_{\mu
u}q_1^\mu q_2^\nu \frac{\epsilon'^* \epsilon'_2}{M_1^2} \frac{1}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \frac{M_k}{2k^2m^2} \frac{(-2M_k^2 + 2k^2m^2)}{(k^2 + M_k^2)^3} \]

\[ \Gamma_2 = \frac{G_F}{M_1^2} \frac{1}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \frac{M_k}{2k^2m^2} \frac{2k^2m^2}{(k^2 + M_k^2)^3} \]

**The sum of 2 integrals after changing variable**

\[ t = \frac{M_1^2(k^2)}{k^2} \Rightarrow \int \frac{dt}{(1 + t)^3} = \frac{1}{2} \]

**and the total contribution is**

\[ \frac{G_F}{M_1^2} \left( \frac{1}{4\pi^2 f_\pi^2} \right) \]

**The result required by Axial Anomaly**

**independent of shape of nonlocality.**

\[ \text{B} \quad \pi_2 \text{ sector} \]

\[ \Gamma_3 = \frac{G_F}{M_1^2} \frac{1}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \frac{M_k}{2k^2m^2} \frac{(-2M_k^2)}{(k^2 + M_k^2)^3} \]

\[ \Gamma_4 = \frac{G_F}{M_1^2} \frac{1}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \frac{M_k}{2k^2m^2} \frac{4k^2m^2}{(k^2 + M_k^2)^3} \]

The sum of these contributions is 0.
Asymptotics of $\tilde{f}_{1,1,1}$ form factor

The leading $Q^2$ behaviour comes from the local part of $\tilde{f}_{1,1,1}$ vertex

$$A^{(1)}(Q^2) \sim \frac{1}{Q^2} \int \frac{d^2 k}{2\pi^2} e^{-iQ^2 k^2 / 2M^2} \frac{1}{(k^2 + M^2)^3} f(k)$$

$\sigma, F$ are Laplace transforms of $\frac{f(k)}{k^2 + M^2}$ and $\frac{1}{k^2 + M^2}$

and with index $m$ for the same functions multiplied by $M^2$.

**Symmetric Case** $q_1^2 = q_2^2 \to 0$

$$A^{(s)}_{\tilde{f}_{1,1,1}}(0, Q^2 \to 0) \sim \frac{1}{Q^2} \frac{\pi q_1^2}{M^2} \int \frac{d^2 k}{2\pi^2} \frac{M^2 (2M^2 - Q^2)}{(Q^2 + M^2)^2} \frac{2}{3}$$

$$= \frac{1}{Q^2} \frac{2}{3} \sigma^{(s)}$$

The similar contribution is from $\tilde{g}_{1,1}^{(c)}$ coupling:

$$A^{(c)}_{\tilde{f}_{1,1,1}}(0, Q^2 \to 0) = \frac{1}{Q^2} \frac{2}{3} \sigma^{(c)}$$

Thus the nonlocal model fixes the absolute normalization of asymptotics.
Consider general asymptotics:

\[ Q^2 = q_1^2 + q_2^2 \to \infty \]
\[ \omega = \frac{q_1^2 - q_2^2}{Q^2} \]

The general asymptotics may be presented in standard form in terms of distribution amplitude:

\[ A_{gg} (0, q_1^2, q_2^2) \sim \frac{1}{Q^2} \int dx \frac{4 \mathcal{F} \left( \frac{x}{2} \right)}{3 \left( 1 - \omega^2 \right) (2x-1)^2} = T(\omega) \]

where

\[ \mathcal{F}^{(n)}(x) = \tilde{g} \int du \int \frac{dA}{2\pi} \left[ (1-x) G_m (u+i\lambda \chi-x) \right] \]
\[ \mathcal{F}^{(0)}(x) = \frac{\tilde{g}}{m} \int du \int \frac{dA}{2\pi} \left[ G_m (u+i\lambda \chi) G_m (u-i\lambda \chi) \right] + \left( x \frac{G_m (u+i\lambda \chi-x) \frac{G_m (u-i\lambda \chi-x)}{x \to x} \right) \]

\[ G(u) = \frac{f(u)}{u + M^2(u)} \quad G_m = \frac{f \cdot M}{u + M^2} \quad G_k = \frac{f \cdot M}{u + M^2} \]

The asymptotic coefficient has been measured by CLEO collab. for the process \( \pi^0 \pi^+ \pi^- \) (\( \omega = 1 \))

\[ T(\omega = 1) = 1.6 \pm 0.3 \quad \text{CLEO } 98' \]

\[ T(\omega = 1) = 1.8 \quad \text{from Model I} \]

\[ T(\omega = 1) = \frac{2}{3} \int_0^1 dx \frac{\mathcal{F}(x)}{x} \]
The predicted shape of DA is very close to prediction QCD SR with nonlocal condensates \( \text{[A.Bakulev, S.Mikhailov]} \).
CLEO data and Predictions

1. CZ distribution amplitude Excluded
2. Local Quark Triangular loop model
   \[ F_{\text{gjg}} \sim \frac{e^2 Q^2}{M^2} \]
   \[ \frac{Q^2}{Q^2 + M^2} \text{ Excluded} \]
3. Nonlocal Quark-phonon model
4. as well as simple VMD parametrization
   \[ F_{gj}(Q^2) = \frac{1}{4m_f} \frac{1}{1 + Q^2/(8m_f^2)} \text{ work well} \]
Conclusions

1. For the first time, the calculations consistent with real photon and high virtualities limits are performed.
2. With effects of π-A mixing being taken into account.
3. The shape of Pion Distribution amplitude and asymptotic coefficient depend on the nonlocal properties of QCD vacuum.
4. It would be of extreme interest if CLEO perform analyses in wider kinematical region.

\begin{align*}
F_{\pi}(Q^2) & \sim \frac{\gamma}{Q^2} \\
\end{align*}

\begin{align*}
M_\pi & \approx 0.3 \text{ GeV} \\
\text{diluted} & \rightarrow \text{dense} \\
\text{vacuum} & 
\end{align*}