TAKING $D \neq 4$ SERIOUSLY: NUMERICAL REAL–VIRTUAL CANCELLATIONS IN MELLIN SPACE

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We propose a method to combine the various, separately divergent, contributions of a differential cross section in a purely numerical way. We show how performing an integral transform allows one to perform a numerical cancellation of the regularized divergences, both in the total cross section and in the differential distributions case, and obtain an accurate and finite result. As a proof of principle, we apply the method to a simple example, $e^+e^- \to q\,\bar{q}$ hard cross section at first order in α_S.

1 Introduction

The Large Hadron Collider (LHC) has started producing pp collisions at 7 TeV centre-of-mass energy, and the experimental collaborations are now performing their first analyses. Their results will be compared to theoretical predictions which have been obtained in the past several years by the phenomenology community, and to others that must still be calculated to even higher order in perturbation theory before they can meet the experimentalists' necessities [1].

The phenomenologists' Grail would be a fully automated method of calculation. However, the available procedures often require extensive work to be adapted to each specific processes, a number of steps in cross section calculations involving non-systematic procedures. For instance, even analytic continuation has to be worked out independently for many different functions, and this limits a method's automatization: the processes it can be applied to can only involve functions whose analytic continuations are already known.

In this work we wish to propose a method that makes it potentially simpler to automate one of the many operations that must be performed in order to evaluate a physical cross section, namely the combination of virtual and real contributions. Our aim will be to perform in a purely numerical way the cancellation of their singularities appearing as poles in powers of $1/\epsilon$ in dimensional regularization. This operation can of course already be performed in a number of different ways, for instance during the integration over phase space using the slicing [2] or the subtraction method [3,4]. However, we speculate that our new method may be fruitfully employed as one of the building blocks of a different approach to automated cross sections calculations.

2 Numerical combination

We consider the combination of the different contributions of a differential hard cross-section: virtual and real terms, and possibly a collinear counter-term. In the standard procedure, this combination is performed by first expanding in power series of ϵ the various terms. A certain

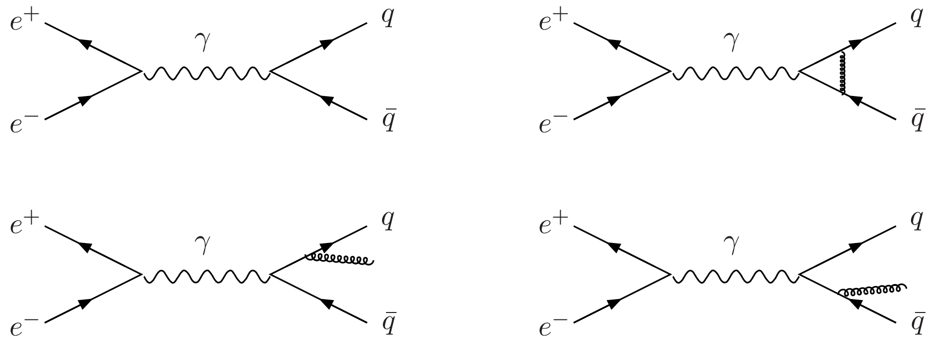

Figure 1: Born (top left), virtual (top right) and real (bottom) Feynman diagrams contributing to the first α_S order of the $e^+e^- \to q\,\bar{q}(g)$ hard cross section.

amount of analytical manipulation is necessary, though it can be automated to a large extent. We propose instead to combine the various contributions in a fully numerical way. In order to do so we use an integral transform, the Mellin one:

$$\mathcal{M}(f)(n) = \tilde{f}(n) = \int_0^1 f(x)x^{n-1}dx \qquad \mathcal{M}^{-1}(\tilde{f})(x) = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-n}\tilde{f}(n)dn \tag{1}$$

The reason for doing so is that the distributions that typically appear in the process of performing the series expansion in powers of ϵ will now be represented by standard functions, susceptible of immediate numerical evaluation. Denoting $\sigma_\epsilon^{(r)}(n)$, $\sigma_\epsilon^{(v)}(n)$ and $\sigma_\epsilon^{(c)}(n)$ the Mellin transforms of the real and virtual parts, and of the collinear counterterm respectively, we can write the full observable cross section as

$$\sigma(n) = \lim_{\epsilon\to 0}\,[\sigma_\epsilon^{(r)}(n) + \sigma_\epsilon^{(v)}(n) + \sigma_\epsilon^{(c)}(n)] \tag{2}$$

where both the sum and the $\epsilon \to 0$ limit will be performed numerically. No series expansion of the separate terms in series of ϵ will be necessary either. One can of course then recover the differential cross section $d\sigma/dx$ by numerical inverse Mellin transform, see eq. (1).

3 Proof of principle: the $e^+e^- \to \gamma^* \to q\,\bar{q}(g)$ hard process

3.1 Differential cross section contributions

As a proof of principle, we investigate the case of the $e^+e^- \to \gamma^* \to q\,\bar{q}(g)$ hard process (Fig 1). More precisely, we consider its one-particle inclusive differential cross-section to order α_S in the Björken variable : $x = 2p.q/q^2$, with p and q being the final state quark and the virtual photon four-momentum respectively.

The full results in dimensional regularization for the the differential cross-section contributions' expressions (real, virtual and $\overline{MS}$ counter-term) in x-space are

$$\begin{aligned}
\frac{d\sigma_\epsilon^{(r)}}{dx} &= \frac{C_F\alpha^2\alpha_S}{W^2}\left(\frac{W^2}{4\pi\mu^2}\right)^{-2\epsilon}\frac{\Gamma(2-\epsilon)^2x^{-2\epsilon}}{(3-2\epsilon)\Gamma(2-2\epsilon)^2}\Big((\epsilon^2-7\epsilon+2)x^2-4\epsilon(\epsilon-2)x+(4\epsilon^2-6\epsilon+2)\Big)\frac{[1-x]^{-1-\epsilon}}{-\epsilon}\\
\frac{d\sigma_\epsilon^{(v)}}{dx} &= \left(\frac{4-2\epsilon}{3-2\epsilon}-2\epsilon\right)\frac{\pi\alpha^2}{W^2}\left[\left(\frac{W^2}{4\pi\mu^2}\right)^{-\epsilon}2^{2\epsilon}\frac{\sqrt{\pi}/2}{\Gamma(\frac{3}{2}-2\epsilon)}\right]\left[1-\left(\frac{W^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{C_F\alpha_S}{\pi}\cos(\epsilon\pi)F_\epsilon\right]\delta(1-x)\\
\frac{d\sigma_\epsilon^{(c)}}{dx} &= \left(\frac{4-2\epsilon}{3-2\epsilon}-2\epsilon\right)\frac{C_F\alpha^2\alpha_S}{2W^2}\left[\left(\frac{W^2}{4\pi\mu^2}\right)^{-\epsilon}2^{2\epsilon}\frac{\sqrt{\pi}/2}{\Gamma(\frac{3}{2}-2\epsilon)}\right]\frac{1}{\Gamma(1-\epsilon)}\frac{1}{\epsilon}\left[\frac{1+x^2}{1-x}\right]_+
\end{aligned} \tag{3}$$

with

$$F_\epsilon = 1 + \Gamma(-\epsilon)\frac{\Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)}\left(\Gamma(-\epsilon) + \frac{1}{2}\Gamma(1-\epsilon)\right) \tag{4}$$

These expressions can be Mellin-transformed exactly, and yield

$$\begin{aligned}
\sigma_\epsilon^{(r)}(n) &= \left(\frac{2-2\epsilon}{3-2\epsilon}\right)\frac{C_F\alpha^2\alpha_S}{W^2}\left[\left(\frac{W^2}{4\pi\mu^2}\right)^{-2\epsilon}\frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)^2}\right]\frac{\Gamma(-\epsilon)^2\;\Gamma(n-2\epsilon)}{\Gamma(2+n-3\epsilon)}\times \\
&\qquad\times\left[2n(1+n)-5(1+\frac{n}{2}(5+n))\epsilon+(21+\frac{n}{2}(25+n))\epsilon^3+8\epsilon^4\right] \\
\sigma_\epsilon^{(v)}(n) &= \left(\frac{4-2\epsilon}{3-2\epsilon}-2\epsilon\right)\frac{\pi\alpha^2}{W^2}\left[\left(\frac{W^2}{4\pi\mu^2}\right)^{-\epsilon}2^{2\epsilon}\frac{\sqrt{\pi}/2}{\Gamma(\frac{3}{2}-2\epsilon)}\right]\left[1-\left(\frac{W^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{C_F\alpha_S}{\pi}\cos(\epsilon\pi)F_\epsilon\right] \\
\sigma_\epsilon^{(c)}(n) &= \left(\frac{4-2\epsilon}{3-2\epsilon}-2\epsilon\right)\frac{C_F\alpha^2\alpha_S}{2W^2}\left[\left(\frac{W^2}{4\pi\mu^2}\right)^{-\epsilon}2^{2\epsilon}\frac{\sqrt{\pi}/2}{\Gamma(3/2-2\epsilon)}\right]\frac{1}{\epsilon\Gamma(1-\epsilon)}\left[-2\sum_{i=1}^{n+1}\frac{1}{i}+\frac{2n+1}{n(n+1)}+\frac{3}{2}\right]
\end{aligned} \tag{5}$$

3.2 Combination

As mentioned above, the standard analytical procedure would consist in expanding each x-space expression in powers of ϵ and summing the results order by order. The singularities must cancel for all the terms with negative powers of ϵ, and one then simply takes the limit for ϵ going to zero of the remainder. The result being finite, this is equivalent to taking the sum of the zeroth order coefficients. The automatization of this procedure is straightforward as long as as one knows the expansions of the different functions of ϵ. However, the appearance of distributions at this stage prevents a straightforward numerical approach.

In n-space, the contributions' expressions are standard functions. This means that we can perform their sum and the limit numerically, without any analytical manipulation. The limit $\epsilon \to 0$ can be evaluated in different ways. Two methods are presented in figure 2, where the cross section as a function of ϵ, $\sigma_\epsilon(n)$, is presented divided by its 'exact' result, i.e. the analytical limit $\sigma(n) = \lim_{\epsilon\to 0}\sigma_\epsilon(n)$ (of course, $\sigma(n)$ can also be thought of as the Mellin transform of the cross section calculated with more standard techniques).

In the left plot, the numerical calculation of $\sigma_\epsilon(n=1)$ has been performed for ϵ values small enough to reach a precision which is sufficient for phenomenology applications. The limit can be approximated by the value of $\sigma_\epsilon(n=1)$ for a small ϵ. In fact, one can see that in a large region $10^{-3} > |\epsilon| > 10^{-6}$ the calculation is numerically stable and its result well within a few per mille of the exact value. In the right plot, the Mellin transforms have been performed numerically in order to simulate a more demanding calculation. In this situation, the numerical precision limits the calculation of $\sigma_\epsilon(n)$ to larger values of $|\epsilon|$, and one cannot anymore approximate the limit by taking ϵ small. In such cases, one can instead use a fit in the region near $\epsilon = 0$ and approximate the limit of $\sigma_\epsilon(n)$ with the value of the fitting function at $\epsilon = 0$. Any precision requirement can be met by properly choosing the value of ϵ in the first method, or the position of the fitted points and the order of the fit in the second. In fact, the first method corresponds to a zeroth order fit.

4 Conclusions and perspectives

We have proposed the use of integral transforms as a way to allow for an easy numerical approach to the problem of combining virtual and real contributions in cross section evaluations. We have shown that the simple case of $e^+e^- \to q\bar{q}(g)$ one-particle inclusive differential hard cross-section

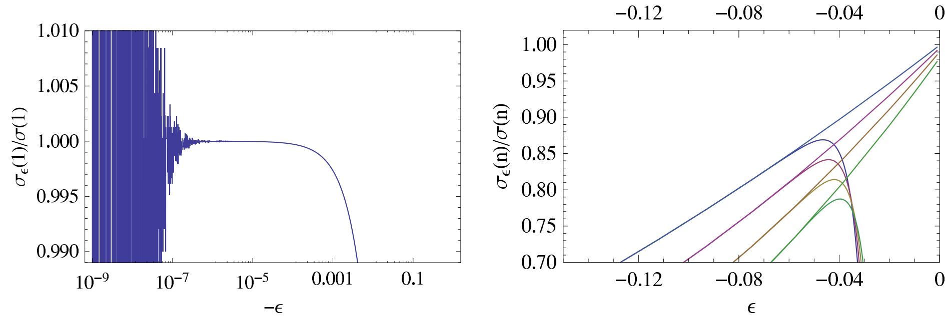

Figure 2: Ratio of the calculated cross section $\sigma_\epsilon(n)$ for $\epsilon \neq 0$ and its exact value for $n = 1$ on the left, and $n = 1$, 10, 10^2 and 10^3 (top to bottom) on the right, as functions of ϵ. In the left plot, $\sigma_\epsilon(1)$ is the sum of the numerical values of the different contributions directly obtained from their analytical expressions in Mellin space. In the right plot, the contributions' values are calculated with a numerical Mellin transform of their analytical expression in x space, and the limits for $\epsilon \to 0$ are evaluated by extrapolating to $\epsilon = 0$ the quadratic fits shown in the plot.

can be treated straightforwardly using a Mellin transform. One advantage of this method is that it does not rely in any way on the knowledge of the soft-collinear limit of the matrix elements: it can therefore used immediately and identically at any order in perturbation theory.

Of course, the burden of the analytical manipulation has been, in first instance, only shifted from the series expansion of the dimensional-regularized expression to the calculation of the Mellin transform. This is straightforwardly done in the simple case that we have treated, but will in general be much harder or even impossible. A possible avenue is to perform the transform numerically, but as we have seen this has a cost in terms of numerical accuracy. An alternative approach is to follow the method proposed in ref. [5], and perform the Mellin transform before the integration over phase space.

Other possible roadblocks concern the extension to more than one variable, in order to obtain fully exclusive distributions. While the problem of performing a multi-dimensional Mellin transform does not appear daunting, their numerical inversion seems instead much more complex than the mono-dimensional case, and we have not yet seriously tackled it. This extension, as well as different approaches to the calculation of the Mellin transforms themselves, will be the subject of future investigations.

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