Taking $D\neq 4$ seriously

Numerical combination of real and virtual contributions in Mellin moment space

Nicolas Houdeau
In collaboration with M. Cacciari

LPTHE, UPMC
General motivation

- Combination of divergencies without previous knowledge of their structure

- Numerical combination to make use of the increasing calculation power
Outline

- Proposed way to perform the combination
- Motivation for the use of Mellin transform
- Illustration with an example as a proof of principle:

\[ e^+ e^- \rightarrow \gamma^* \rightarrow q \bar{q} (g) \]
Calculation of a (partonic) differential cross section

Given a theory (Standard Model for example)
Calculation of a (partonic) differential cross section

One calculates the squared S-matrix amplitude of the process.

$|\mathcal{M}|^2$
One then chooses an observable $X$, function of the momenta,

$$|\mathcal{M}|^2 \delta(x - X(\{p\}))$$
Calculation of a (partonic) differential cross section

And integrates over the whole phase-space (in D=4-2\(\varepsilon\) dimensions)

\[
\int |\mathcal{M}|^2 \delta(x - X(\{p\})) \, d^D\phi
\]
Thus obtaining one part of the differential cross section

\[ \int |M^{(i)}|^2 \delta(x - X(\{p\})) \, d^D \phi = \frac{d\sigma^{(i)}_\epsilon}{dX}(x) \]
The cross section is the sum of the real and virtual parts, and possibly the collinear counter terms.

\[
\frac{d\sigma^{(r)}}{dX} + \frac{d\sigma^{(v)}}{dX} + \frac{d\sigma^{(c)}}{dX} = \frac{d\sigma}{dX} \quad \& \text{cancellation}
\]
The divergencies are removed analytically by expanding the expressions in \( \epsilon \), canceling the poles and then taking the 0-limit.

\[
\frac{d\sigma^{(r)}_{\epsilon}}{dX} + \frac{d\sigma^{(v)}_{\epsilon}}{dX} + \frac{d\sigma^{(c)}_{\epsilon}}{dX} = \frac{d\sigma_{\epsilon}}{dX} \quad \text{& cancellation}
\]
We propose to perform these sums and cancellation, numerically, in Mellin moment space.

\[
\sigma^{(r)}_\epsilon(n) + \sigma^{(v)}_\epsilon(n) + \sigma^{(c)}_\epsilon(n) = \sigma_\epsilon(n) \quad \& \text{cancellation}
\]
UV, IR and collinear divergencies are regularized in dimensional regularization:

\[
\frac{d\sigma^{(i)}_{\epsilon}}{dX} = \ldots \frac{1}{\epsilon^2} + \ldots \frac{1}{\epsilon} + \ldots + \mathcal{O}(\epsilon)
\]

Distributions in x are present in the series in powers of \(\epsilon\):

\[
\frac{d\sigma^{(i)}_{\epsilon}}{dX} = \ldots \left( \frac{1}{1-x} \right) + \frac{1}{\epsilon^2} + \ldots \frac{1}{\epsilon} + \ldots \delta(1-x) + \mathcal{O}(\epsilon)
\]

They can arise either from the phase space integrals, or from the \(\epsilon\) expansion, in the distributional sense, of functions of x.
Cancellation of divergencies
Advantages of Mellin transform

- **x space:**
- UV, IR and collinear divergencies are regularized in dimensional regularization:

\[
\frac{d\sigma^{(i)}}{dX} = \left( \frac{1}{1-x} + \frac{1}{\epsilon^2} + \frac{1}{\epsilon} + \ldots \right) + \mathcal{O}(\epsilon)
\]

- Distributions in x are present in the series in powers of \( \epsilon \):

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\frac{d\sigma^{(i)}}{dX} = \left( \frac{1}{1-x} + \frac{1}{\epsilon^2} + \frac{1}{\epsilon} + \ldots \right) + \mathcal{O}(\epsilon)
\]

Numerical approach difficult

They can arise either from the phase space integrals, or from the \( \epsilon \) expansion, in the distributional sense, of functions of x.
Mellin space:

Mellin transform is an integral transform:

\[ M(f)(n) = \int_0^1 x^{n-1} f(x) \, dx \]

Where \( n \) is the Mellin moment (\( n \) is a complex number)

\( x \) distributions are represented by functions of \( n \)

Mellin transform is adapted to QCD calculations: convolutions appearing in calculations become products in Mellin space

No more distribution

Only numerical quantities
**An example:** \( e^+ e^- \rightarrow \gamma^* \rightarrow q \bar{q} (g) \)

- **Starting point:**

Differential cross section contributions' expression in Björken variable: \( X = \frac{2p \cdot q}{q^2} \), \( q \) and \( p \), the CM and quark moment (\( W^2 = q^2 \)).

- Virtual and real contributions:

\[
\frac{d\sigma^{(v)}_\epsilon}{dX}(x) = \left( \frac{4 - 2\epsilon}{3 - 2\epsilon} - 2\epsilon \right) \pi \alpha^2 \left( \frac{W^2}{4\pi \mu^2} \right)^{-\epsilon} \frac{2^2\epsilon}{\Gamma(3/2 - 2\epsilon)} \Gamma(2 - 2\epsilon) \right] \frac{1 - \left( \frac{W^2}{4\pi \mu^2} \right)^{-\epsilon} \frac{C_F \alpha_S}{\pi} \cos(\epsilon\pi) F_\epsilon}{\delta(1 - x)} 
\]

\[
\frac{d\sigma^{(r)}_\epsilon}{dX}(x) = \frac{C_F \alpha^2 \alpha_S}{W^2} \left( \frac{W^2}{4\pi \mu^2} \right)^{-2\epsilon} \frac{1}{\Gamma(3 - 2\epsilon)} \Gamma(2 - 2\epsilon) \right] x^{-2\epsilon} \left( (\epsilon^2 - 7\epsilon + 2)x^2 - 4\epsilon(\epsilon - 2)x + (4\epsilon^2 - 6\epsilon + 2) \right) \delta(1 - x) 
\]

- Collinear counter term in \( \overline{\text{MS}} \) factorization scheme:

\[
\frac{d\sigma^{(c)}_\epsilon}{dX}(x) = \left( \frac{4 - 2\epsilon}{3 - 2\epsilon} - 2\epsilon \right) \frac{C_F \alpha^2 \alpha_S}{2W^2} \left[ \left( \frac{W^2}{4\pi \mu^2} \right)^{-\epsilon} \frac{\sqrt{\pi}/2}{\Gamma(3/2 - 2\epsilon)} \right] \frac{1}{\Gamma(1 - \epsilon)} \frac{1}{\epsilon} \left[ \frac{1 + x^2}{1 - x} \right]_+ 
\]
An example: \( e^+ e^- \rightarrow \gamma^* \rightarrow q \bar{q} (g) \)

Starting point:

Differential cross section contributions' expression in Björken variable: \( X = \frac{2p \cdot q}{q^2} \), q and p, the CM and quark moment (\( W^2=q^2 \)).

Virtual and real contributions:

Collinear counter term in \( \overline{\text{MS}} \) factorization scheme:

\[
\frac{d\sigma^{(e)}_{\epsilon}}{dX}(x) = \frac{C_F}{W} \left( \frac{4 - 2\epsilon}{3 - 2\epsilon} - 2\epsilon \right) C_F \alpha_s^2 \frac{W^2}{4 \pi \mu^2} \left[ \left( \frac{W^2}{\Gamma(3/2 - 2\epsilon)} \right)^{2\epsilon} \frac{\sqrt{\pi}/2}{\Gamma(3/2 - 2\epsilon)} \right] \frac{1}{\Gamma(1 - \epsilon)} \frac{1}{\epsilon} \left[ \frac{1 + x^2}{1 - x} \right]_+ 
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Differential cross section contributions' expression in Björken variable: \( X = \frac{2p \cdot q}{q^2} \), \( q \) and \( p \), the CM and quark moment \((W^2=q^2)\).

- **Virtual and real contributions:**

\[
\frac{d\sigma^{(v)}_\epsilon}{dX}(x) = \left( \frac{4-2\epsilon}{3-2\epsilon} - 2\epsilon \right) \frac{\pi \alpha^2}{W^2} \left[ \left( \frac{W^2}{4\pi \mu^2} \right)^{-\epsilon} \left( \frac{\sqrt{\pi/2}}{\Gamma(3/2-2\epsilon)} \right) \right] \left[ 1 - \left( \frac{W^2}{4\pi \mu^2} \right)^{-\epsilon} \frac{C_F \alpha_S}{\pi} \cos(\epsilon \pi) F_\epsilon \right] \delta(1-x)
\]

\[
\frac{d\sigma^{(r)}_\epsilon}{dX}(x) = \frac{C_F \alpha^2 \alpha_S}{W^2} \left( \frac{W^2}{4\pi \mu^2} \right)^{-2\epsilon} \frac{\Gamma(2-\epsilon)^2}{(3-2\epsilon)\Gamma(2-2\epsilon)^2} x^{-2\epsilon}((\epsilon^2 - 7\epsilon + 2) x^2 - 4\epsilon(\epsilon - 2)x + (4\epsilon^2 - 6\epsilon + 2)) \cdot [\epsilon]^{-1} [1-x]^{-1-\epsilon}
\]

- **Collinear counter term in \( \overline{\text{MS}} \) factorization scheme:**

\[
\frac{d\sigma^{(c)}_\epsilon}{dX}(x) = \left( \frac{4-2\epsilon}{3-2\epsilon} - 2\epsilon \right) \frac{C_F \alpha^2 \alpha_S}{2W^2} \left[ \left( \frac{W^2}{4\pi \mu^2} \right)^{-\epsilon} \frac{\sqrt{\pi/2}}{\Gamma(3/2-2\epsilon)} \right] \frac{1}{\Gamma(1-\epsilon)} \frac{1}{\epsilon} \left[ \frac{1+x^2}{1-x} \right]_+
\]
We perform an $\epsilon$ expansion:

\[
\frac{d\sigma^{(v)}_\epsilon}{dX}(x) = \left( \frac{4 - 2\epsilon}{3 - 2\epsilon - 2\epsilon} - \frac{2\epsilon}{W^2} \right) \frac{\pi \alpha^2}{4\pi \mu^2} \left[ \left( \frac{W^2}{4\pi \mu^2} \right)^{-\epsilon} \frac{2^{2\epsilon}}{\Gamma(3/2 - 2\epsilon)} \left[ 1 - \left( \frac{W^2}{4\pi \mu^2} \right)^{-\epsilon} \frac{C_F \alpha_S}{\pi} \cos(\epsilon \pi) F_\epsilon \right] \delta(1 - x) 
\]

\[
\frac{d\sigma^{(r)}_\epsilon}{dX}(x) = \frac{C_F \alpha^2 \alpha_S}{W^2} \left( \frac{W^2}{4\pi \mu^2} \right)^{-2\epsilon} \frac{\Gamma(2 - \epsilon)^2}{(3 - 2\epsilon) \Gamma(2 - 2\epsilon)^2} x^{2\epsilon} \left( (\epsilon^2 - 7\epsilon + 2)x^2 - 4\epsilon(\epsilon - 2)x + (4\epsilon^2 - 6\epsilon + 2) \right) \cdot \left[ -\epsilon \right]^{-1} \left[ 1 - x \right]^{-1-\epsilon}
\]

\[
\frac{d\sigma^{(c)}_\epsilon}{dX}(x) = \left( \frac{4 - 2\epsilon}{3 - 2\epsilon - 2\epsilon} - \frac{2\epsilon}{W^2} \right) \frac{C_F \alpha^2 \alpha_S}{2W^2} \left[ \left( \frac{W^2}{4\pi \mu^2} \right)^{-\epsilon} \frac{2^{2\epsilon}}{\Gamma(3/2 - 2\epsilon)} \frac{\sqrt{\pi}}{\Gamma(3/2 - 2\epsilon)} \right] \left[ 1 - \epsilon \right] \left[ \frac{1 + x^2}{1 - x} \right]_+ 
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\]

\[
\frac{d\sigma^{(r)}_\epsilon}{dX}(x) = \frac{C_F \alpha^2 \alpha_S}{W^2} \left(\frac{W^2}{4\pi \mu^2}\right)^{-2\epsilon} \frac{\Gamma(2 - \epsilon)^2}{(3 - 2\epsilon)\Gamma(2 - 2\epsilon)^2} x^{-2\epsilon} \left((\epsilon^2 - 7\epsilon + 2)x^2 - 4\epsilon(\epsilon - 2)x + (4\epsilon^2 - 6\epsilon + 2)\right) [-\epsilon]^{-1} [1 - x]^{-1 - \epsilon}
\]

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\]
We perform an $\epsilon$ expansion:

\[
\frac{d\sigma^{(v)}_\epsilon}{dX}(x) = \frac{4}{3} \frac{C_F \alpha^2 \alpha_S}{W^2} \left( -\delta(1-x) \frac{1}{\epsilon^2} + \left(2 \log \left( \frac{W^2}{4\pi \mu^2} \right) + 2\gamma - \frac{13}{6}\right) \delta(1-x) \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \right)
\]

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\frac{d\sigma^{(r)}_\epsilon}{dX}(x) = \frac{C_F \alpha^2 \alpha_S}{W^2} \left( \frac{W^2}{4\pi \mu^2} \right)^{-2\epsilon} \frac{\Gamma(2-\epsilon)^2}{(3-2\epsilon)\Gamma(2-2\epsilon)^2} x^{-2\epsilon} ((\epsilon^2 - 7\epsilon + 2)x^2 - 4\epsilon(\epsilon - 2)x + (4\epsilon^2 - 6\epsilon + 2)) \cdot [\epsilon]^{-1} [1-x]^{-1-\epsilon}
\]

\[
\frac{d\sigma^{(c)}_\epsilon}{dX}(x) = \frac{4}{3} \frac{C_F \alpha^2 \alpha_S}{W^2} \left( \frac{1}{2} \left( \frac{1+x^2}{1-x} \right) + \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0) \right)
\]
Analytical combination (x space)

We perform an $\varepsilon$ expansion:

$$
\frac{d\sigma^{(v)}_{\varepsilon}}{dX}(x) = \frac{4}{3} \frac{C_F \alpha^2 \alpha_S}{W^2} \left( -\delta(1-x) \frac{1}{\varepsilon^2} + (2 \log \left( \frac{W^2}{4\pi \mu^2} \right) + 2\gamma - \frac{13}{6} ) \delta(1-x) \frac{1}{\varepsilon} + O(\varepsilon^0) \right)
$$

$$
\frac{d\sigma^{(r)}_{\varepsilon}}{dX}(x) = \frac{C_F \alpha^2 \alpha_S}{W^2} \left( \frac{W^2}{4\pi \mu^2} \right)^{-2\varepsilon} \frac{\Gamma(2-\varepsilon)^2}{(3-2\varepsilon)\Gamma(2-2\varepsilon)^2} x^{-2\varepsilon} ((\varepsilon^2 - 7\varepsilon + 2) x^2 - 4\varepsilon(\varepsilon - 2)x + (4\varepsilon^2 - 6\varepsilon + 2)) \left[ -\varepsilon^{-1} [1-x]^{-1-\varepsilon} \right]
$$

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\frac{d\sigma^{(c)}_{\varepsilon}}{dX}(x) = \frac{4}{3} \frac{C_F \alpha^2 \alpha_S}{W^2} \left( \frac{1}{2} \left( \frac{1+x^2}{1-x} \right) + \frac{1}{\varepsilon} + O(\varepsilon^0) \right)
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\]

\[
\frac{d\sigma^{(c)}_{\varepsilon}}{dX}(x) = \frac{4}{3} \frac{C_F \alpha^2 \alpha_S}{W^2} \left( \frac{1}{2} \left( \frac{1 + x^2}{1 - x} \right) \right) \frac{1}{\varepsilon} + O(\varepsilon^0)
\]

\[
[-\varepsilon]^{-1} \left[ 1 - x \right]^{-1 - \varepsilon} = \delta(1-x) \frac{1}{\varepsilon^2} - \left( \frac{1}{1-x} \right) \frac{1}{\varepsilon} + \left( \frac{\log(1-x)}{1-x} \right) + O(\varepsilon)
\]
We perform an $\varepsilon$ expansion:

$$\frac{d\sigma_{\varepsilon}^{(v)}}{dX}(x) = \frac{4}{3} C_F \alpha^2 \alpha_S \frac{W^2}{W^2} \left( -\delta(1 - x) \frac{1}{\varepsilon^2} + \left( 2 \log \left( \frac{W^2}{4 \pi \mu^2} \right) + 2\gamma - \frac{13}{6} \right) \delta(1 - x) \frac{1}{\varepsilon} + O(\varepsilon^0) \right)$$

$$\frac{d\sigma_{\varepsilon}^{(r)}}{dX}(x) = \frac{4}{3} C_F \alpha^2 \alpha_S \frac{W^2}{W^2} \left( \delta(1 - x) \frac{1}{\varepsilon^2} + \left[ -\frac{1}{2} \left( \frac{1 + x^2}{1 - x} \right) + \left( 2 \log \left( \frac{W^2}{4 \pi \mu^2} \right) + 2\gamma - \frac{13}{6} \right) \delta(1 - x) \frac{1}{\varepsilon} + O(\varepsilon^0) \right)$$

$$\frac{d\sigma_{\varepsilon}^{(c)}}{dX}(x) = \frac{4}{3} C_F \alpha^2 \alpha_S \frac{W^2}{W^2} \left( \frac{1}{2} \left( \frac{1 + x^2}{1 - x} \right) + \frac{1}{\varepsilon} + O(\varepsilon^0) \right)$$

We sum the contributions and take the limit: $\varepsilon \rightarrow 0$

These are the manipulations we propose, here, to avoid...
Mellin space

Instead of performing the expansion, we Mellin-transform the different contributions:

\[
\sigma^{(v)}_\epsilon(n) = \left( \frac{4 - 2\epsilon}{3 - 2\epsilon} - 2\epsilon \right) \frac{\pi \alpha^2}{W^2} \left[ \left( \frac{W^2}{4\pi \mu^2} \right)^{-\epsilon} 2^{2\epsilon} \frac{\sqrt{\pi}/2}{\Gamma(3/2 - 2\epsilon)} \right] \left[ 1 - \left( \frac{W^2}{4\pi \mu^2} \right)^{-\epsilon} \frac{C_F \alpha_S}{\pi} \cos(\epsilon \pi) F_\epsilon \right]
\]

\[
\sigma^{(r)}_\epsilon(n) = \left( \frac{2 - 2\epsilon}{3 - 2\epsilon} \right) \frac{C_F \alpha^2 \alpha_S}{W^2} \left[ \left( \frac{W^2}{4\pi \mu^2} \right)^{-2\epsilon} \frac{\Gamma(2 - \epsilon)}{\Gamma(2 - 2\epsilon)^2} \right] \Gamma(-\epsilon)^2 f(n, \epsilon)
\]

\[
f(n, \epsilon) = \frac{\Gamma(n - 2\epsilon)}{\Gamma(2 + n - 3\epsilon)} \left[ 2n(1 + n) - 5(1 + \frac{n}{2}(5 + n))\epsilon + (21 + \frac{n}{2}(25 + n))\epsilon^3 + 8\epsilon^4 \right]
\]

\[
\sigma^{(c)}_\epsilon(n) = \left( \frac{4 - 2\epsilon}{3 - 2\epsilon} - 2\epsilon \right) \frac{C_F \alpha^2 \alpha_S}{2W^2} \left[ \left( \frac{W^2}{4\pi \mu^2} \right)^{-\epsilon} 2^{2\epsilon} \frac{\sqrt{\pi}/2}{\Gamma(3/2 - 2\epsilon)} \right] \frac{1}{\Gamma(1 - \epsilon)} \frac{1}{\epsilon} \left( -2 \sum_{i=1}^{n+1} \frac{1}{i} + \frac{1}{n} + \frac{1}{n+1} + \frac{3}{2} \right)
\]

Those are now purely numerical quantities: we can manipulate them without resorting to an \(\epsilon\) expansion.
We can approximate $\sigma(n)$ by $\sigma_\varepsilon(n)$ with $\varepsilon$ sufficiently small.

$$\varepsilon = 10^{-2} : \frac{\sigma_\varepsilon(1)}{\sigma(1)} = 0.973527$$

$$\varepsilon = 10^{-5} : \frac{\sigma_\varepsilon(1)}{\sigma(1)} = 0.999973$$

The sum of the three contributions, $\sigma_\varepsilon(n)$, has a finite limit, $\sigma(n)=M(d\sigma/dX)$.

It tends smoothly to this limit since it has only positive powers of $\varepsilon$ in its Taylor serie.
We can approximate \( \sigma(n) \) by \( \sigma(\epsilon(n)) \) with \( \epsilon \) sufficiently small. It tends smoothly to this limit since it has only positive powers of \( \epsilon \) in its Taylor serie.

Note that one cannot take \( \epsilon \) infinitely small: the two contributions diverge, and the numerical cancellation becomes unreliable when \( \epsilon \) goes to zero, \( n \) fixed, \( \sigma^{(r)} + \sigma^{(c)} \) has the same divergent behavior than \( -\sigma^{(v)} \).

The sum of the three contributions, \( \sigma^{(n)} \), has a finite limit, \( \sigma(n) = \text{M}(d\sigma/dX) \).

**Numerical combination (Mellin space)**

![Graph showing numerical combination in Mellin space](image)
To simulate a more demanding numerical calculation, we perform the Mellin transform numerically.

The cancellation fails much earlier, at $\epsilon \approx 5 \times 10^{-2}$. 
Numerical combination (Mellin space)

To simulate a more demanding numerical calculation, we perform the Mellin transform numerically.

We can adapt the order of the fit to the precision needed.

linear fit : \( \frac{\sigma_{fit}(1)}{\sigma(1)} = 0.980333 \)

quadratic fit : \( \frac{\sigma_{fit}(1)}{\sigma(1)} = 0.999255 \)

\( \epsilon = -10^{-1} ; -0.9 \times 10^{-1} ; -0.8 \times 10^{-1} \)

The cancellation fails much earlier, at \( \epsilon \approx 5 \times 10^{-2} \).

But we use our knowledge on the behavior of the sum \( \sigma_{\epsilon}(n) \) for small \( \epsilon \).

We fit it around \( \epsilon = 0 \) (here, quadratically).
Results in x-space

- $\sigma(n)$ contains all the information of $d\sigma/dX$. We usually would perform subsequent calculations in Mellin space.

- We can still verify our results by performing the inverse Mellin transform directly on $\sigma(n)$ and compare it to the regular part of $d\sigma/dX$.

$\sigma(n)$ was calculated with a quadratical fit with:

$$\epsilon = -10^{-1} ; -0.9 \times 10^{-1} ; -0.8 \times 10^{-1}$$
**Conclusion**

- No manipulation of the divergences in the combination
- Numerical method that can be adapted to any precision requirement
- Note: use of the $\varepsilon$ dependency in Mellin space
- Alternative way to perform the combination of virtual and real contributions
- Still, this is only a first step. We would need to:
  - Apply the method on more exclusives and/or physically more relevant cross sections
  - Test/develop compatible methods for the previous and successive operations: phase-space integration and, inverse Mellin transform
**Extensions and limits**

Calculation of the differential cross sections directly in Mellin space (e.g. A. Mitov, *Phys. Lett. B643* (2006) 366-373)
Extensions and limits

Calculation of the differential cross sections directly in Mellin space (e.g. A. Mitov, *Phys. Lett. B643* (2006) 366-373)

\[
\int_0^1 x^{n-1} |\mathcal{M}|^2 \delta(x - X(p)) \, dx = |\mathcal{M}|^2 X(p)^{n-1}
\]
**Extensions and limits**

- Calculation of the differential cross sections directly in Mellin space (e.g. A. Mitov, *Phys. Lett. B* 643 (2006) 366-373)

- Generalization to more exclusive cross sections: the limits imposed by the inverse transform

- Analytical continuation in $n$ variable (in the $\text{Re}[n]<0$ half plane)

\[
M^{-1}(\tilde{f})(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-n} \tilde{f}(n) \, dn
\]
Extensions and limits

Calculation of the differential cross sections directly in Mellin space (e.g. A. Mitov, Phys. Lett. B643 (2006) 366-373)

Generalization to more exclusive cross sections: the limits imposed by the inverse transform

Analytical continuation in n variable (in the Re[n]<0 half plane)

\[ M^{-1}(\tilde{f})(x) = \frac{1}{2\pi i} \int_{C} x^{-n(s)} \tilde{f}(n(s)) J(s) \, ds \]
Extensions and limits

- Calculation of the differential cross sections directly in Mellin space (e.g. A. Mitov, *Phys.Lett. B643 (2006) 366-373*)

- Generalization to more exclusive cross sections: the limits imposed by the inverse transform
  - Analytical continuation in n variable (in the Re[n]<0 half plane)

$$M^{-1}(\tilde{f})(x) = \frac{1}{2\pi i} \oint_{C} x^{-n(s)} \tilde{f}(n(s)) \ J(s) \ ds$$

![Diagram showing Mellin space with contours for analytical continuation]
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Extensions and limits

- Calculation of the differential cross sections directly in Mellin space (e.g. A. Mitov, *Phys.Lett. B643* (2006) 366-373)

- Generalization to more exclusive cross sections: the limits imposed by the inverse transform
  - Analytical continuation in n variable (in the Re[n]<0 half plane)
  - Integration contour for the inverse Mellin transform in more than one complex dimension

- Necessity of the ε dependancy