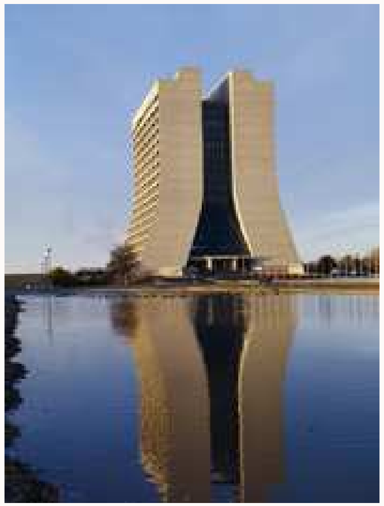
Calculation of QCD loops using tree-level matrix elements

[Rencontres de Moriond, QCD & High-Energy Interactions, La Thuile]

Jan Winter [a]

– Fermilab –



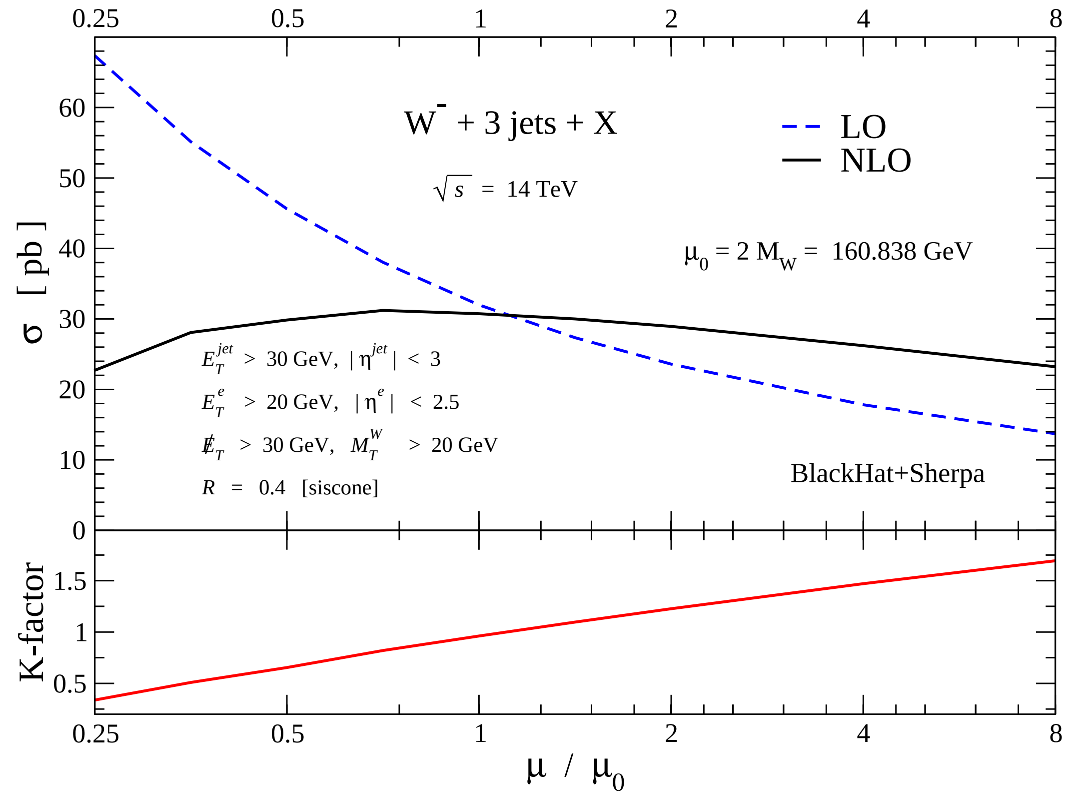
- *Few words on NLO calculations*
- *Generalized unitarity and computation of one-loop amplitudes*
- *Numerical method based on colour-dressed amplitudes*
- *Results for multiple gluon scattering*

[a] In collaboration with: W. Giele and Z. Kunszt

Need of higher-order calculations

➡ ***Lessons learned from LEP, HERA, Tevatron:***
LO predictions are fine, yet often only give rough estimates

e.g. NLO: 1st real prediction of normalization of many observables
less sensitivity to unphysical input scales (μ_F & μ_R)
more physics (parton merging, jet substructure, ISR, more IS parton species)



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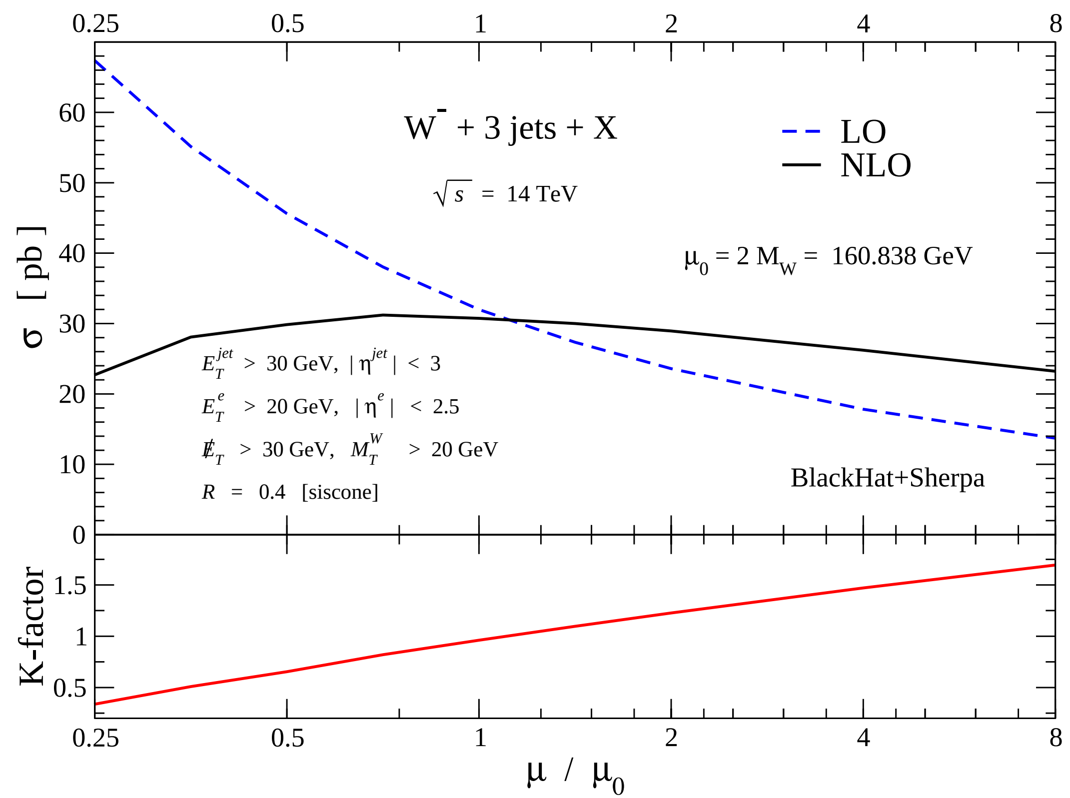
➡ ***Components of NLO calculations***

- tree-level amplitudes (LO & real radiation)
- \+ one-loop correction to Born level
- \+ subtraction terms to handle and combine singularities
- \+ phase-space generator



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- tree-level a
  (LO & real
- \+ one-loop c
- \+ subtraction
  combine si
- \+ phase-spac

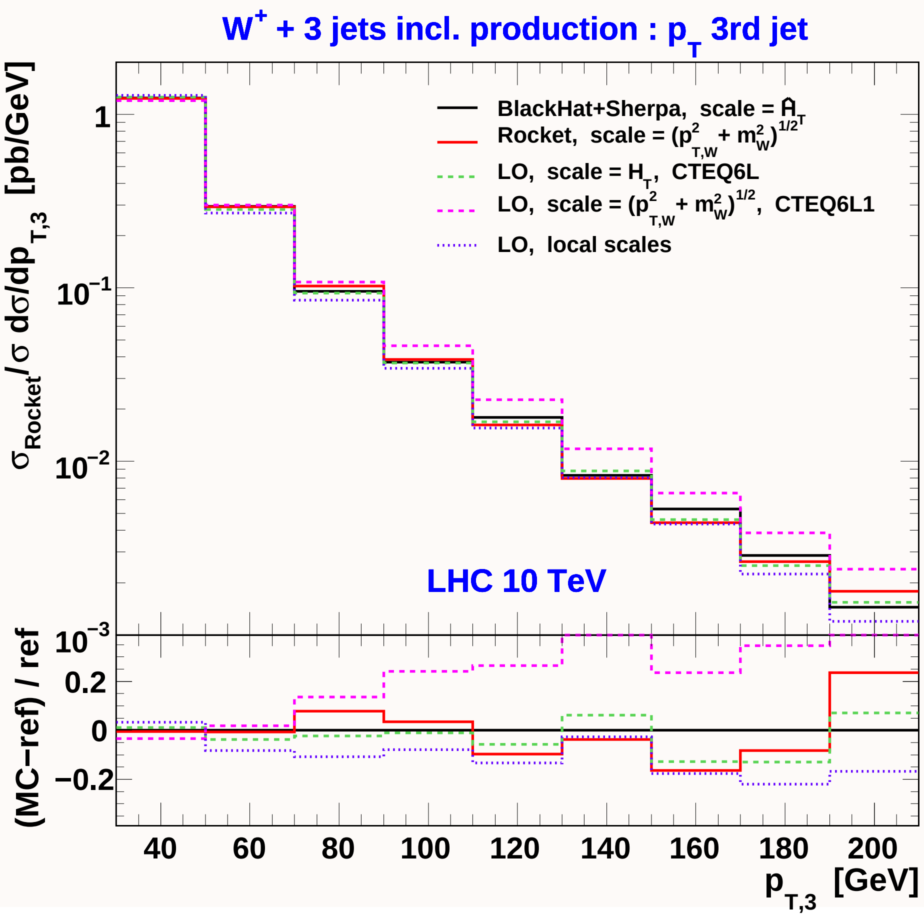
➡ for example, BLACKHAT+SHERPA

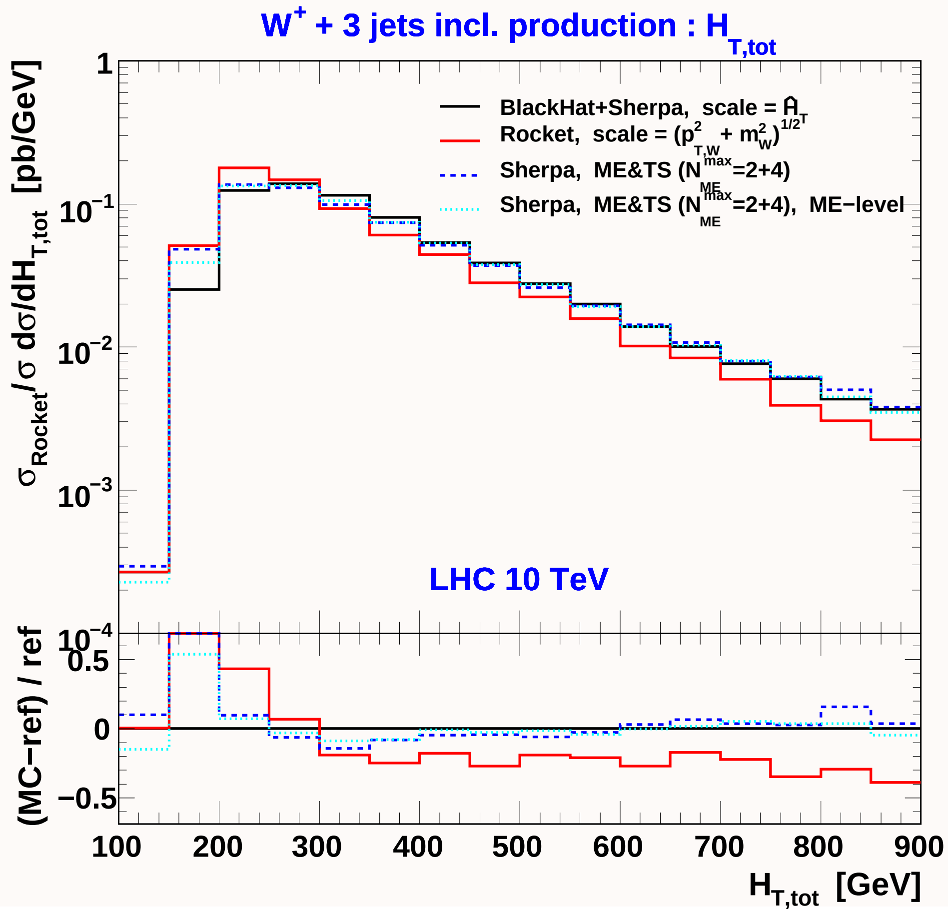
$$\sigma = \int_m d\sigma^B + \int_{m+1} (d\sigma^R - d\sigma^A) + \int_m (d\sigma^V + \int_1 d\sigma^A)$$

A recent comparison of LHC predictions for W+3jets

[Höche, Huston, Maitre, Winter, Zanderighi; LH09 proceed.: arXiv:1003.1241]

- between BlackHat [Berger et al.], Rocket [Ellis, Melnikov, Zanderighi] and Sherpa [Gleisberg et al.]
- rather different scale choices at NLO yield > 20% deviations … impact on BSM searches !
- Sherpa's ME&TS merging in good agreement with NLO once rescaled to NLO xsec
- certain scale choices at LO can mimic NLO results … often strongly depends on observable





Virtual correction and colour decomposition

$$d\sigma_{\rm V}(f_1 f_2 \to f_3 \ldots f_N) \;\sim\; \int d\Phi(p_1 \ldots p_N)\; 2\,{\rm Re}\Big(\mathcal{M}^{(0)}(f_1 \ldots f_N)^* \times \mathcal{M}^{(1)}(f_1 \ldots f_N)\Big)$$

➡ ***factorization of one-loop amplitude in colour factors and primitive amplitudes is systematic***

- colour decomposition of one-loop N-gluon amplitude in $SU(N_{\rm C})$ gauge theory

$$\mathcal{M}^{(1)} \;=\; g^N \sum_{\sigma \in S_{N-1}/\mathcal{R}} {\rm Tr}(F^{a_{\sigma_1}} \cdots F^{a_{\sigma_N}})\; \mathcal{A}_N^{(1)[1]}(\sigma_1, \ldots, \sigma_N) \quad + \quad 2\,n_f\; g^N \sum_{\sigma \in S_{N-1}/\mathcal{R}} {\rm Tr}(\lambda^{a_{\sigma_1}} \cdots \lambda^{a_{\sigma_N}})\; \mathcal{A}_N^{(1)[1/2]}(\sigma_1, \ldots, \sigma_N)$$

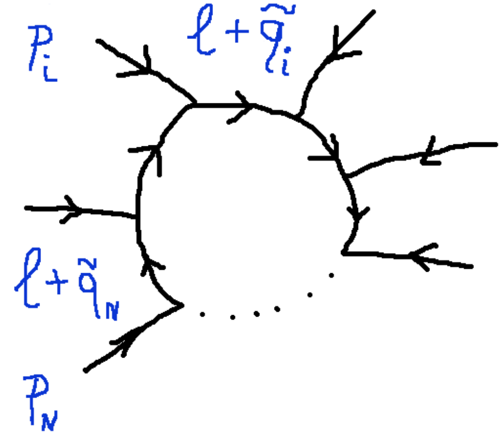
  allows for separate treatment of colour factors and primitive or ordered amplitudes

- N gluons, only ask for leading-colour contributions ... make use of phase-space symmetry

$$\int d\Phi\;{\rm Re}(\mathcal{M}^{(0)*}\mathcal{M}^{(1)}) \;\sim\; \sum_{\rm perm} \int d\Phi\;{\rm Re}(\mathcal{A}^{(0)*}\mathcal{A}^{(1)}) \;\approx\; (N-1)! \int d\Phi\;{\rm Re}(\mathcal{A}^{(0)*}\mathcal{A}^{(1)})$$

➡ simplifications come in handy when calculating a specific process (both BlackHat and Rocket use these tricks) — however not optimal for automation

Decomposition of one-loop amplitudes



$$\mathcal{A}_N^{(1)}(\{p_i\}) \;=\; \int \frac{d^D\ell}{i\pi^{D/2}} \frac{\mathcal{N}(\{p_i\}\mid\ell)}{d_{i_1} d_{i_2} \cdots d_{i_N}} \,, \qquad d_i(\ell) = (\ell+\tilde{q}_i)^2 - m_i^2$$

➡ ***decompose into a linear sum of scalar box, triangle, bubble and tadpole master integrals (cut-constructible part) and rational terms***

$$\mathcal{A}_N^{(1)}(\{p_i\}) \;=\; \sum_{[i_1|i_4]} d_{i_1 i_2 i_3 i_4} I^{(D)}_{i_1 i_2 i_3 i_4} + \sum_{[i_1|i_3]} c_{i_1 i_2 i_3} I^{(D)}_{i_1 i_2 i_3} + \sum_{[i_1|i_2]} b_{i_1 i_2} I^{(D)}_{i_1 i_2} + \sum_{[i_1|i_1]} a_{i_1} I^{(D)}_{i_1} + \mathcal{R}_N$$

- master integrals known in literature
- and implemented in various codes, e.g. QCDLoop [Ellis, Zanderighi] (QCDLoop.fnal.gov)
- To do: determination of the master-integral coefficients

  ⬅ generalized-unitarity techniques [Britto, Cachazo, Feng — Bern, Dixon, Dunbar, Kosower]

  ⬅ subtraction terms to extract lower-point coefficients best identified at the integrand level [Ossola, Papadopoulos, Pittau]

note that $[i_1|i_m] = 1 \le i_1 < i_2 < \ldots < i_M \le N$ and $I^{(D)}_{i_1 \ldots i_M} = \int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{d_{i_1} \cdots d_{i_M}}$

Basics of Ellis–Giele–Kunszt–Melnikov method

- integrand is re-expressed by sum of basic denominator structures

$$\mathcal{A}_N^{(1)}(\{p_i\} \mid \ell) = \sum_{k=1}^{5} \sum_{[i_1|i_k]} \frac{\mathcal{P}(\vec{c}_{i_1\ldots i_k} \mid \ell)}{d_{i_1}\cdots d_{i_k}}$$

- numerators encode ℓ dependence ➡ parametric form: polynomial functions in coefficients

$$\mathcal{P}(\vec{c}_{i_1\ldots i_k} \mid \ell) \sim \sum_j \alpha_j(\ell) \times c^{(j)}_{i_1\ldots i_k} = \text{MI} + \text{rational} + \text{spurious} \quad \text{terms}$$

$\int d^D\ell \;\ldots$ MI terms $= c^{(0)}_{i_1\ldots i_k} I_{i_1\ldots i_k}$ and rational terms $= c^{(r)}_{i_1\ldots i_k}/\#$

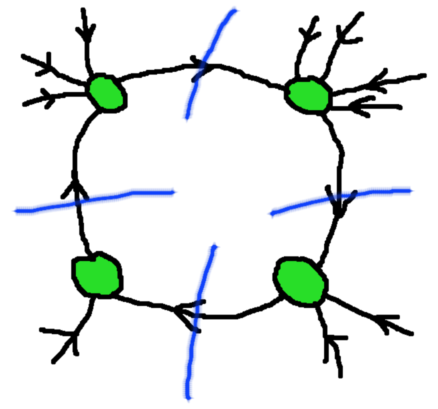
spurious terms vanish upon integration

- solve for coefficients by solving systems of equations given by $\ell = \tilde{\ell}$ such that $d_{i_1},\ldots,d_{i_n} \equiv 0$

$$\mathcal{P}(\vec{c}_{i_1\ldots i_n} \mid \tilde{\ell}) = \sum_{\text{dof}}^{\text{internal}} \prod_{k=1}^{n} \mathcal{M}^{(0)}\left(\tilde{\ell}_{i_k}, \{p_j\}, -\tilde{\ell}_{i_{k+1}}\right)$$

$$- \sum_{k=n+1}^{5} \sum_{[i_1|i_k]} d_{i_1}\cdots d_{i_n} \frac{\mathcal{P}(\vec{c}_{i_1\ldots i_k} \mid \tilde{\ell})}{d_{i_1}\cdots d_{i_k}}$$

Using tree-level MEs !



Algorithm for full one-loop amplitudes

- ***EGKM implementations to calculate ordered amplitudes are robust and sufficiently fast***

  using Berends–Giele recursion relations to determine the $\mathcal{M}^{(0)}$ pieces yields algorithm of polynomial complexity $(\tau \sim N^{\#})$ [GIELE, ZANDERIGHI — LAZOPOULOS — GIELE, WINTER]

- In general, the sum over colour orderings has to be performed in some way $\Rightarrow$ obtain $\mathbf{2\,Re}(\boldsymbol{\mathcal{M}}^{(0)*}\boldsymbol{\mathcal{M}}^{(1)})$ **...** ***may become laborious*** ... all orderings need be known

- (naive) permutation sum re-introduces factorial growth ... $(N-1)!/2$
- complexity of colour decomposition increases for quark dominated processes

- Can we do better ... tame the growth ?

  Construction of an algorithm of exponential complexity. Colour quantum #s included.

  $\Rightarrow$ Naive expectation of the asymptotic scaling is $(f \times 5)^N$ for N legs.

  $\Rightarrow$ Colour-dressed recursions give factor $f > 1$, can be as large as 4.

  $\Rightarrow$ Number of pentagons rise with 5^N ... asymptotic behaviour of Stirling # $\mathcal{S}_2(N, 5)$.

input: external parton momenta & polarizations plus their explicit colours (colour-flow representation)
output: amplitude $\mathcal{M}^{(1)}$ in the form of a complex number

EGKM extended

[GIELE, KUNSZT, WINTER, ARXIV:0911.1962]

➡ ***Start off the EGKM algorithm for colour-ordered amplitudes. To include full colour information, extensions are necessary:***

- Decomposition of the integrand: sums over ordered cuts change into sums over partitions including non-cyclic, non-reflective permutations of the initial partitions.

$$\sum_{[i_1|i_k]} \quad \rightarrow \quad \sum_{RP_{\pi_1\ldots\pi_k}(1,2,\ldots,N)}$$

- Identification of the subtraction terms when solving for $\mathcal{P}(\vec{c}_{\pi_1\ldots\pi_k} \mid \tilde{\ell})$: identify by de-pinching, account for possible shifts in loop momenta.

- Calculation of the integrand's residues: use colour-dressed recursions and sum over internal polarizations and internal colours.

$$\sum_{\text{dof}}^{\text{internal}} \prod_{k=1}^{n} \mathcal{M}^{(0)}\left(\tilde{\ell}_{i_k}, \{p_j\}, -\tilde{\ell}_{i_{k+1}}\right) \quad \rightarrow \quad \sum_{\substack{\{\lambda_j\} \\ \{(IJ)_j\}}} \prod_{k=1}^{n} \mathcal{M}^{(0)}\left(\tilde{\ell}_{\pi_k}^{(\lambda_k (IJ)_k)}, p_{\pi_k}, -\tilde{\ell}_{\pi_{k+1}}^{(\lambda_{k+1}(JI)_{k+1})}\right)$$

- Decomposition of one-loop amplitude: comes with symmetry factor of $1/2!$ in front of the bubble-coefficient terms.

Unordered gluons: a note on partitions

- number of unitarity cuts, example 4-gluon loop

  ordered) 0|1|2|3
  01|2|3, 0|12|3, 0|1|23, 1|2|30
  0|123, 1|230, 2|301, 3|012, 01|23, 12|30

  unordered) 0|1|2|3, 0|2|3|1, 0|3|1|2
  0|1|23, 0|2|13, 0|3|12, 1|2|03, 1|3|02, 2|3|01
  01|23, 02|13, 03|12

ord.)	N	5-gons	boxes	triangles	bubbles	total
3	4	0	1	4	6	11
12	5	1	5	10	10	26
60	6	6	15	20	15	56
360	7	21	35	35	21	112
2520	8	56	70	56	28	210
20160	9	126	126	84	36	372

unord.) N	5-gons	boxes	triangles	bubbles	total
4	0	3	6	3	12
5	12	30	25	10	77
6	180	195	90	25	490
7	1680	1050	301	56	3087
8	12600	5103	966	119	18788
9	83412	23310	3025	246	109993

ord.) number of orderings however grows as $(N-1)!/2$, unord.) Stirling numbers grow as k^N

- number of k-cut combinations: $\mathcal{C}(N,k) = \binom{N}{k}$ but to multiply with number of orderings
- number of $k \geq 2$-cut partitions: $\max\{1, (k-1)!/2\} \times \mathcal{S}_2(N,k) - N\,\Theta(2-k)$
  $\Rightarrow$ increased number of terms, origin of exponential growth

Scaling behaviour of the algorithm

- Table taken from an early test: $2 \to N-2$ gluons

$(++--..)$ polarizations, $\binom{..1131..}{..1311..}$ colours & random PSPs obeying separation cuts ...

computation times in secs (2.20 GHz Intel Core2 Duo)

ord.)	N	cut-c,4D	factor	full,5D	factor	unord.) N	cut-c,4D	factor	full,5D	factor	OK?
2	4	0.025		0.045		4	0.05		0.105		✓
6	5	0.185	7.4	0.355	7.9	5	0.315	6.3	0.74	7.0	✓
24	6	0.83	4.5	2.7	7.6	6	1.37	4.3	4.59	6.2	✓
120	7	7.95	9.6	27.5	10.2	7	8.4	6.1	32.5	7.1	✓
720	8	86.5	10.9	328	11.9	8	52	6.2	234	7.2	✓
5040	9	1070	12.4	4250	13.0	9	354	6.8	1720	7.4	✓
40320	10	14000	13.1	60600	14.3	10			13700	8.0	✓

ord.) factors clearly increase with larger N, unord.) growth follows $(f \cdot 5)^N$, $1 < f < 2$

number of non-zero colour factors grows as $(N-2)!$ for this case

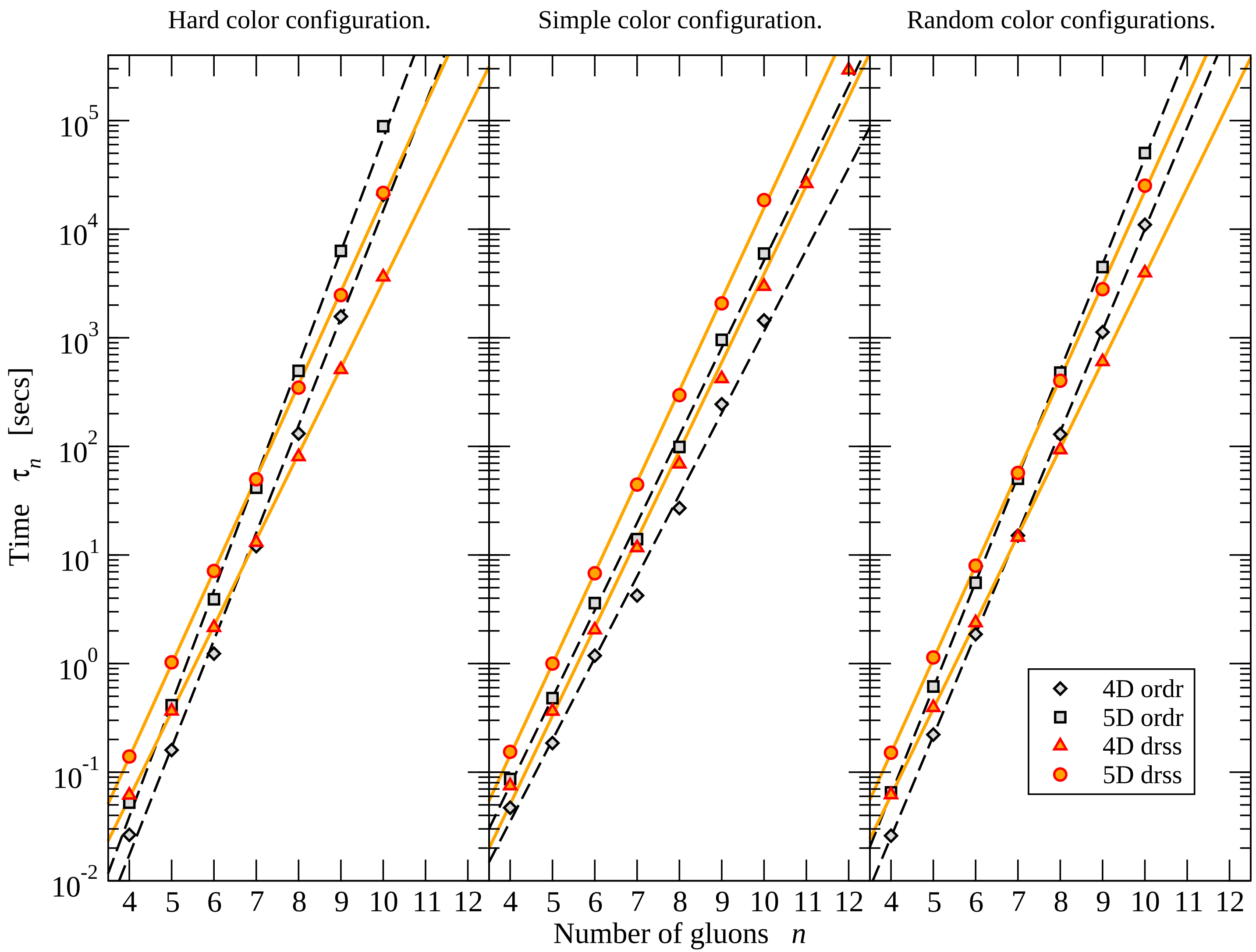
Scaling of the computation time with # of legs

(calculations in double precision) [Giele, Kunszt, Winter, arXiv:0911.1962]

- algorithm checked for exponential complexity ($\tau \sim x^N$)

- Random colours:

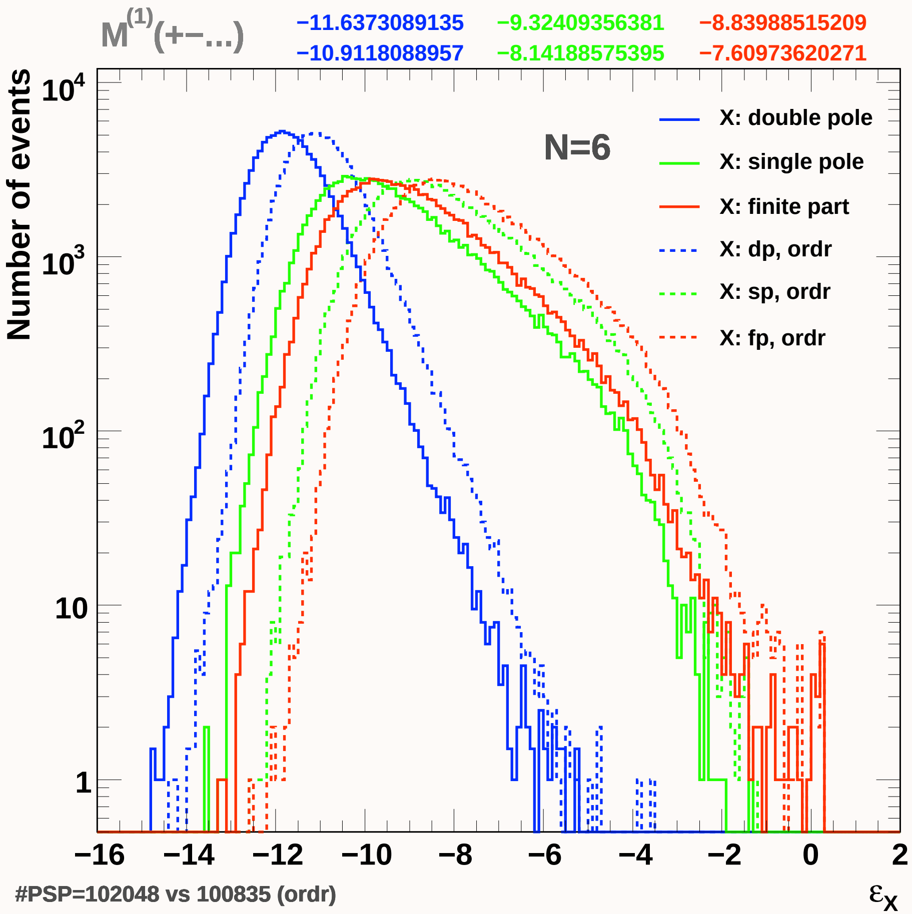
Algorithm	x
4dim ordered	8.6(1)
5dim ordered	9.5(1)
4dim unord.	6.30(4)
5dim unord.	7.3(1)

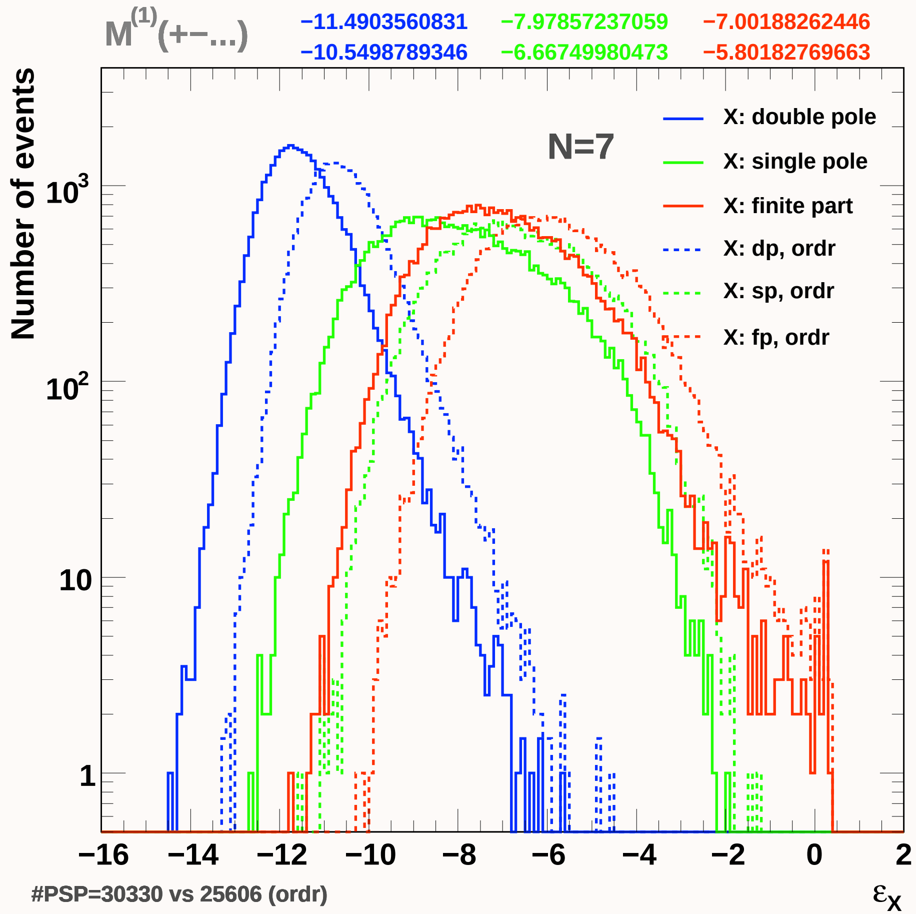


Accuracy

(calculations in double precision) [Giele, Kunszt, Winter, arXiv:0911.1962]

- peak positions & tails are OK, 97% ($N = 6$) and 89% (unord.) vs. 96% and 87% (ord.) of events can be handled with double precision
- unordered algorithm provides on average more accurate results





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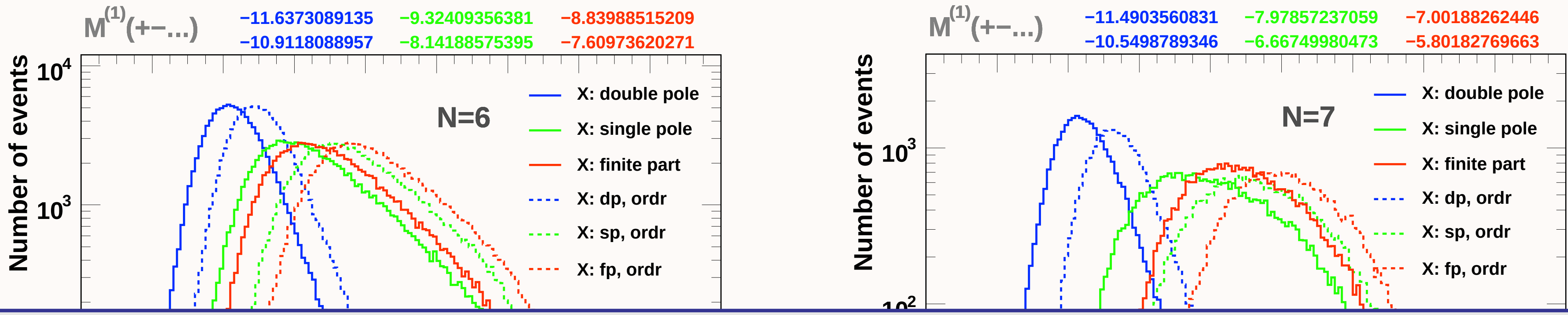


- accuracy — numerical stability of algorithm

$$\varepsilon_{\rm dp} = \log_{10} \frac{|\mathcal{M}^{(1)[1]}_{\rm dp,num} - \mathcal{M}^{(1)}_{\rm dp,th}|}{|\mathcal{M}^{(1)}_{\rm dp,th}|}, \qquad \varepsilon_{\rm s/fp} = \log_{10} \frac{2\,|\mathcal{M}^{(1)[1]}_{\rm s/fp,num} - \mathcal{M}^{(1)[2]}_{\rm s/fp,num}|}{|\mathcal{M}^{(1)[1]}_{\rm s/fp,num}| + |\mathcal{M}^{(1)[2]}_{\rm s/fp,num}|}$$

Phase-space integration and colour sampling tests

(calculations in double precision) [GIELE, KUNSZT, WINTER, ARXIV:0911.1962]

- stability & consistency check: test convergence of uniform phase-space Monte Carlo integrations
- colour sampled:

  $S_{\mathrm{MC}} = W_{\mathrm{col}} \times \mathcal{K}$

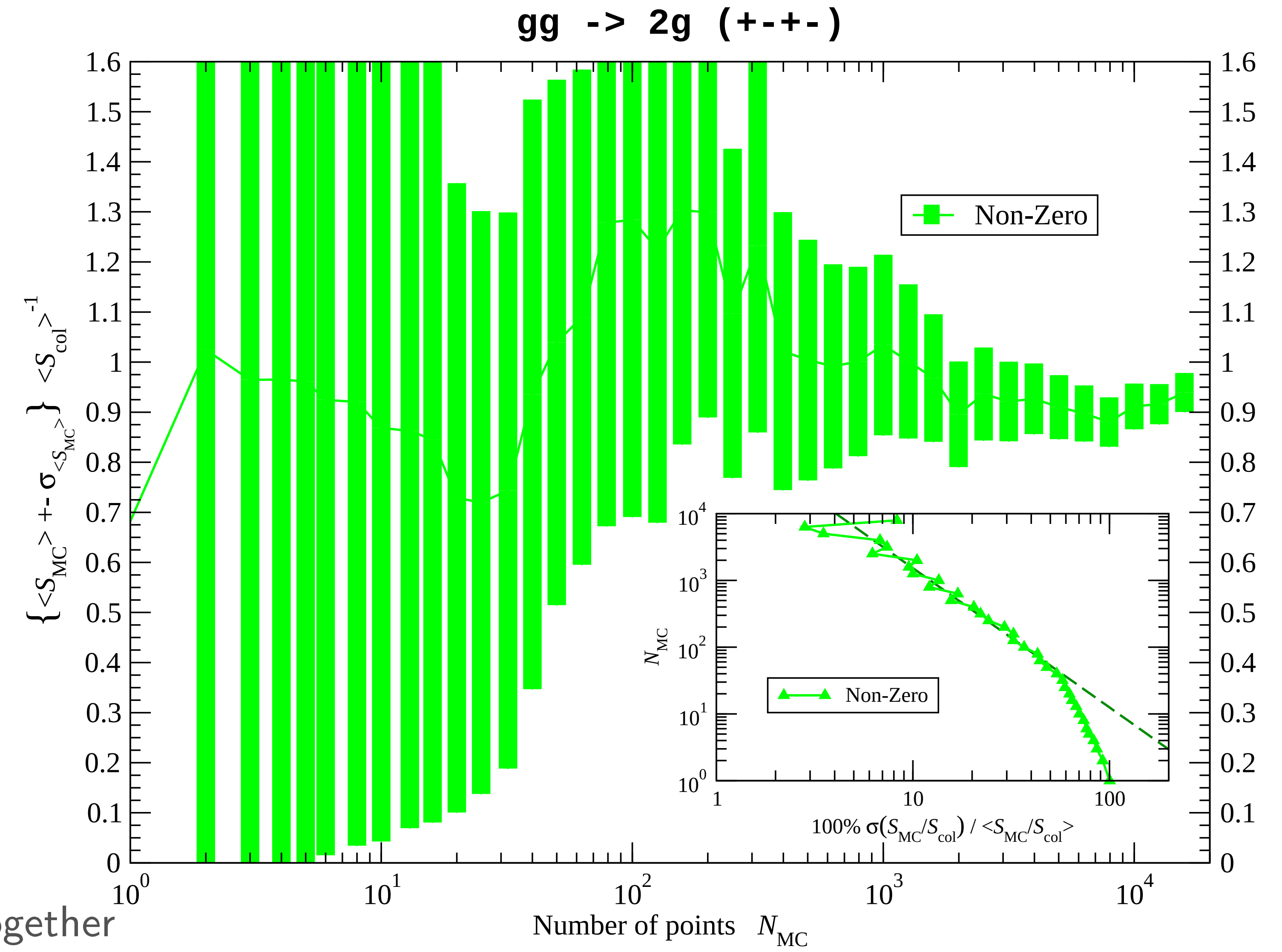
  normalized to colour summed:

  $S_{\mathrm{col}} = \sum_{\mathrm{col}} \mathcal{K}$

  with the kernel

  $\mathcal{K} = |\mathcal{M}^{(0)}| + \frac{\alpha_s}{2\pi} \times \mathrm{Re}(\mathcal{M}^{(1)}_{\mathrm{fp}} \mathcal{M}^{(0)\dagger})$

- only display standard deviation of $\langle S_{\mathrm{MC}} \rangle$ ➡ MC phase-space integration and colour sampling work together



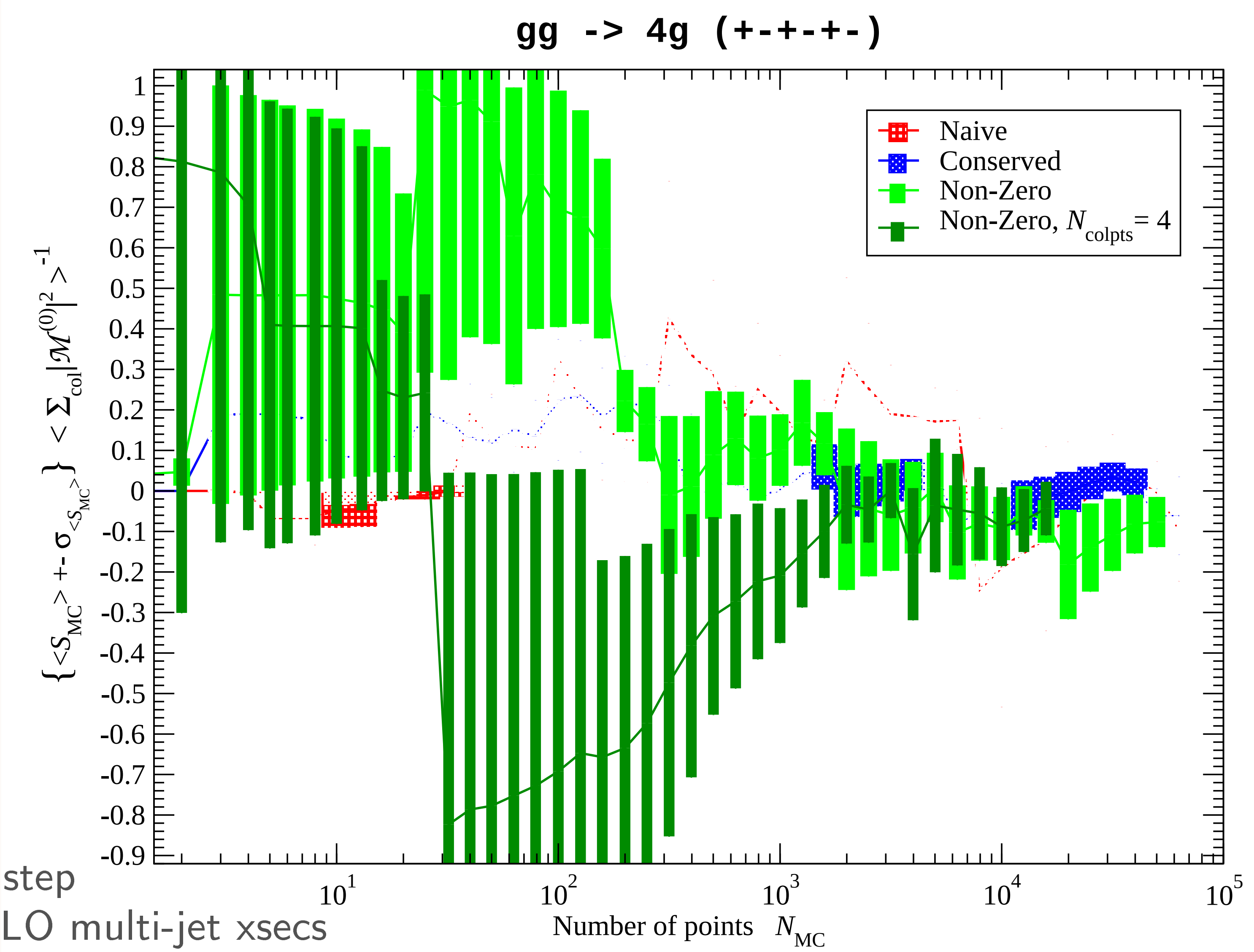
Phase-space integration and colour sampling tests

(calculations in double precision) [Giele, Kunszt, Winter, arXiv:0911.1962]

- stability check: test convergence of virtual corrections when integrated over a flat phase space
- colour sampled:

  $S_{\rm MC} = W_{\rm col} \times \mathcal{K}$

  normalized to colour summed Born contribution
- good estimate of magnitude of virtual correction
- different sampling schemes ⇒ $W_{\rm col}$
- only display standard deviation of $\langle S_{\rm MC} \rangle$
- first test of one major step in the calculation of NLO multi-jet xsecs



Outlook : construction of an NLO generator

➡ ***COMIX ... SM tree-level ME generator based on colour-dressed generalized BG recursions*** [Gleisberg, Höche]

- calculates colour-dressed tree-level amplitudes
- exponential growth

➡ ***Use as LO generator in the numerical generalized-unitarity method***

- gives colour-dressed one-loop amplitudes for gluons, quarks, ..., basically the whole SM
- truly "automated" generation of virtual corrections seems feasible

➡ full NLO MC for arbitrary SM processes when augmented by a phase-space and bremsstrahlung generator

- e.g. combining it with automated Catani–Seymour subtraction of Gleisberg & Krauss ...

Summary

- Higher-order calculations are needed to meet the requirements on the precision of theoretical predictions in the LHC era.

- Highly automated and optimized parton-level event generators are available at tree level. At one loop, similar achievements seem possible owing to the new methods based on generalized unitarity and parametric integration techniques that use tree-level amplitudes as their input.

- Calculations based on recursive methods are easier to automate.
  Presented recursive scheme for the computation of QCD one-loop amplitudes that incorporates colour along with all other degrees of freedom.
  - $\Rightarrow$ algorithm is an extension of the Ellis–Giele–Kunszt–Melnikov method.

- Algorithmic implementation for full amplitudes using colour-dressed recursion relations.
  - $\Rightarrow$ algorithm is of exponential complexity.
  - $\Rightarrow$ asymptotic scaling of $\sim 7^N..8^N$ seen — milder than for colour decomposition.
  - $\Rightarrow$ more to do: fully include quarks, squared amplitudes, OLE, xsecs (pure jets).
  - $\Rightarrow$ potential improvements: fitting coefficients, higher precision.

- Numerical results presented for colour-dressed one-loop gluon amplitudes. Algorithm works.
  - $\Rightarrow$ reasonably accurate double-precision results — more accurate than for colour decomposition.
  - $\Rightarrow$ colour-sampling convergence tested when integrating $2\mathrm{Re}(\mathcal{M}^{(0)*}\mathcal{M}^{(1)})$ over phase space ✓