Multi-loop integrals made simple: applications to QCD processes

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Towards automating NNLO calculations

- status scattering amplitudes & cross sections at NLO
  - efficient tools for generating & organizing integrands
  - one-loop integrals known
  - subtraction methods available
  - many automated programs

- at NNLO: bottleneck often missing analytic expressions for loop integrals

- this talk: new method for computing loop integrals
  - based on better understanding Feynman integrands and integrals
  - identifies appropriate class of functions (and computes them!)
  - makes analytic properties manifest (e.g. singularities)
  - especially useful for integrals that depend on several scales
  - for massive/massless, planar/non-planar integrals
  - uses differential equations

References:
- [JMH, PRL 110 (2013) 25]
- Gehrmann and Remiddi, NPB 580 (2000) 485]
Sample applications: two-scale problems

• massless 2-2 scattering to 3 loops

\[ s = (p_1 + p_2)^2 \quad t = (p_2 + p_3)^2 \]
\[ x = t/s \]

non-planar integrals work in the same way

[JMH, A.V. Smirnov, V.A. Smirnov, 2013]
JHEP 1307 (2013) 128

physics motivation: scattering amplitudes in Yang-Mills & supergravity

• heavy quark effective theory

all 3-loop cusp integrals, e.g.

\[ \cos \phi = \frac{v_1 \cdot v_2}{\sqrt{v_1^2} \sqrt{v_2^2}} , \quad x = e^{i\phi} \]

physics motivation: infrared divergences of massive scattering amplitudes
Sample applications: multi-scale problems

- integrals for Bhabha scattering
  \[ \text{[J.M.H., V. Smirnov, JHEP 1311 (2013) 041]} \]

- scattering amplitudes & cross sections in massive toy model
  \[ \text{[JMH, S. Caron-Huot, to appear] \hspace{1cm} s, t, m^2} \]

\[ \text{3 loops and 3 scales!} \]

similar integrals in QCD for finite top quark mass

- vector boson production \( pp \rightarrow VV \)
  \[ \text{[JMH, Melnikov, V. Smirnov, 1402.7078]} \]

\[ \frac{S}{M_3^2} = (1 + x)(1 + xy), \quad \frac{T}{M_3^2} = -xz, \quad \frac{M_4^2}{M_3^2} = x^2y \]

equal mass case: \[ \text{[Gehrmann, Tancredi, Weihs, JHEP 1308 (1013) 070]} \]
Key points of the method

- differential equations for master integrals $\vec{f}$
- crucial: choose convenient basis (systematic procedure) $\rightarrow$ makes solution trivial to obtain
- elegant description: Feynman integrals specified by:
  1. set of ‘letters’ (related to singularities $x_k$)
  2. set of constant matrices $A_k$

Example: one dimensionless variable $x$; $D = 4 - 2\epsilon$

$$\partial_x \vec{f}(x; \epsilon) = \epsilon \sum_k \frac{A_k}{x - x_k} \vec{f}(x; \epsilon)$$

- expansion to any order in $\epsilon$ is linear algebra
- answer: multiple polylogarithms of uniform weight (‘transcendentality’)
- asymptotic behavior $\vec{f}(x; \epsilon) \sim (x - x_k)^\epsilon A_k \vec{f}_0(\epsilon)$
- natural extension to multi-variable case
vector boson production \[ pp \rightarrow VV \]

- planar integral families

\[
\frac{S}{M_3^2} = (1 + x)(1 + xy), \quad \frac{T}{M_3^2} = -xz, \quad \frac{M_4^2}{M_3^2} = x^2 y
\]

physical region \( 0 < x, \quad 0 < y < z < 1 \)

- differential equations

\[
df(x, y, z; \epsilon) = \epsilon d \tilde{A}(x, y, z) \cdot f(x, y, z; \epsilon)
\]

\[
\tilde{A} = \sum_{i=1}^{15} \tilde{A}_{\alpha_i} \log(\alpha_i)
\]

- alphabet \( \alpha = \{ x, y, z, 1 + x, 1 - y, 1 - z, 1 + xy, z - y, 1 + y(1 + x) - z, xy + z, 1 + x(1 + y - z), 1 + xz, 1 + y - z, z + x(z - y) + xyz, z - y + yz + xyz \} \).

- solution in terms of multiple polylogarithms
Conclusions and outlook

• families of Feynman integrals described by iterated integrals
  analytic answer specified by:
  (1) “alphabet” for iterated integrals
  (2) constant matrices provide rules to form words
• many applications for LHC physics, e.g.
  - amplitudes involving top quarks
  - vector boson production
  - … (insert your wish here)
  - outlook: library for NNLO Feynman integrals
• interesting mathematics
  - Fuchsian differential equations; monodromies, asymptotic limits
  - extension to elliptic functions
Curious?

lecture notes for this method available in chapter 3.8 of