On Regge theory and Lorentzian OPE in CFTs

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This note resumes the relation between the Lorentzian OPE limit in a Conformal Field Theory and the Regge limit of four-point functions. Both kinematical limits are controlled by the leading twist operators. We explain how conformal Regge theory can be used to relate the dimension of the spin $J$ leading twist operators to the spin of the Reggeon that dominates t-channel exchange in the Regge limit, as well as to extract non-trivial information about the OPE coefficients between external operators and the exchange leading twist operators.

Regge theory is a technique to study high energy scattering. It is particularly useful when an infinite number of resonances participate in a scattering process, like in QCD or in string theory. Since conformal gauge theories are equivalent to string theory in Anti-de Sitter (AdS) space, it is natural to look for a generalization of Regge theory to scattering in AdS, which could be a good starting point to understand AdS/CFT correlation functions in the stringy regime of finite string tension (or finite 't Hooft coupling), beyond the gravity approximation.

In this note we resume some recent results on the Pomeron-graviton Regge trajectory in maximally supersymmetric Yang-Mills theory (SYM). One of the great values of Regge theory is that the relation between the Reggeon spin and dimension of the exchanged operators does not commute with perturbation theory. Furthermore, we shall see that the same statement applies for the relation between Regge residue and OPE coefficient between the external operators and the exchanged operator of spin $J$.

The general expectation for the behaviour of the leading Regge trajectory in a CFT is depicted in figure 1. The leading Regge trajectory is the set of operators of lowest dimension for each spin $J$, also known has leading-twist operators. This set of operators is important in two interesting kinematical limits, for which the four-point function $\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4) \rangle$ is essentially reduced to a function of a single variable. The first limit, which we call Lorentzian OPE, corresponds to making $x_{12} = x_1 - x_2$ approach the lightcone. In this limit, the OPE is dominated by the operators with lowest twist $\tau = \Delta - J$, where $\Delta$ is the conformal dimension and $J$ the spin. This limit is relevant for phenomenological applications like Deep Inelastic Scattering.
The second kinematical limit is the Regge limit, which basically corresponds to performing a large boost on points $x_1$ and $x_2$. Naively, this limit is dominated by the operators with maximal spin. However, since there are operators with arbitrarily large spin, a more careful analysis requires summing the contributions from an infinite number of operators with increasing spin. This has been done in $^4,^5$ using Regge theory methods that involve the analytic continuation of the leading Regge trajectory $\Delta(J)$ to complex values of spin.

In a conformal field theory the Regge limit of a four-point function is obtained from a specific Lorentzian kinematical limit where all the points are taken to null infinity $^4,^6$. This limit can be defined by $x_1^+ \rightarrow \lambda x_1^+$, $x_2^+ \rightarrow \lambda x_2^+$, $x_3^+ \rightarrow \lambda x_3^+$, $x_4^- \rightarrow \lambda x_4^-$ and $\lambda \rightarrow \infty$, keeping the causal relations $x_{14}^2, x_{23}^2 < 0$. We will choose all the other $x_{ij}^2 > 0$, as show in figure 2, although this is not essential. This Lorentzian regime, needed to take the Regge limit, can be obtained from the Euclidean regime by analytic continuation. This is done by fixing $\bar{z}$ and by analytically continuing in $z$ counter clockwise around 0 and 1, as also shown in figure 2. Physical space-time points now correspond to both $z$ and $\bar{z}$ real numbers. It is then convenient to introduce variables $\sigma$ and $\rho$ related to the usual cross ratios $u$ and $v$ by

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \equiv \sigma^2, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \equiv (1 - \sigma e^\rho)(1 - \sigma e^{-\rho}) \approx 1 - 2\sigma \cosh \rho. \quad (1)$$

The Regge limit corresponds now to $\sigma \rightarrow 0$ with fixed $\rho$.

The connection between the the Lorentzian OPE limite and the Regge limit can be established through Conformal Regge Theory $^5$. The starting point is the conformal block decomposition of the four-point function written as

$$A(u, v) = \sum_{J=0}^{\infty} \int_{-\infty}^{\infty} d\nu \frac{i\nu}{\pi K_{2+\nu,J}} d_J(\nu^2) G_{2+\nu,J}(u, v), \quad (2)$$

where

$$K_{\Delta,J} = \frac{\Gamma(\Delta + J) \Gamma(\Delta - 1 + J)}{4^{J-1} \Gamma^6 \left(\frac{\Delta + J}{2}\right) \Gamma^2 \left(\frac{\Delta - J}{2}\right)}. \quad (3)$$

To reproduce the exchange of an operator of dimension $\Delta$ and spin $J$ the partial amplitude must have the singular behaviour

$$d_J(\nu^2) \approx a_{\Delta,J} \frac{K_{\Delta,J}}{\nu^2 + (\Delta - 2)^2}. \quad (4)$$

The next step is to consider the Sommerfeld-Watson transform, keeping only the operators in the leading Regge trajectory, for which we have a relation $\Delta = \Delta(J) = 2 + J + \gamma(J)$. Analytically continuing in the spin $J$, the Sommerfeld-Watson integration over $J$ can be done by picking the Regge pole $j(\nu)$ defined by

$$\nu^2 + \left(\Delta(j(\nu)) - 2\right)^2 = 0, \quad (5)$$
leading to the final result

\[ A(\sigma, \rho) \approx 2\pi i \int d\nu \alpha(\nu) \sigma^{1-j(\nu)} \Omega_{\nu}(\rho), \]  

where

\[ \alpha(\nu) = -\frac{\pi 2^2 j(\nu - 3) e^{\pi j(\nu)/2}}{\nu \sin\left(\frac{\pi j(\nu)}{2}\right)} \Gamma^2\left(\frac{2 + j(\nu) + i\nu}{2}\right) \Gamma^2\left(\frac{2 + j(-\nu) - i\nu}{2}\right) j'(\nu) K_{2+i\nu,j(\nu)} b_{j(\nu)}, \]

with \( b_j \) defined by the product of OPE coefficients. We refer to the function \( \alpha(\nu) \) as the Regge residue, since it is related to the residue of the dominant Regge pole that follows from (4).

Equations (5) and (7) establish a precise relation between the two kinematical limits described above. The pomeron spin \( j(\nu) \) and residue \( \alpha(\nu) \) are related to analytic continuations of the dimensions \( \Delta(J) \) and (square of) OPE coefficients \( b_j \) of the leading twist operators. Since relation (5) does not commute with perturbation theory it is possible to obtain all-loop predictions for the function \( \Delta(J) \) from the BFKL spin \( j(\nu) \), and vice versa. For SYM this was explored at weak coupling in \(^3\) and at strong coupling in \(^5\). Thus, both eqs. (5) and (7) can be used as a consistency check on the available data in both regimes. In particular, they also provide a new test that direct computations of OPE coefficients must pass.

As an example let us consider the relation between spin and anomalous dimension of the operators in the Pomeron Regge trajectory at strong coupling in SYM, which actually describes the graviton Regge trajectory in \( \text{AdS}_5 \) space. The anomalous dimensions of the leading twist operators can be computed from the energy of short strings in AdS, and admit the expansion \(^7, 8\)

\[ \Delta(J)(\Delta(J) - 4) = x \left[ 2\sqrt{\lambda} + \left( -1 + \frac{3x}{2} \right) - \frac{3}{8} \left( -10 + (8\zeta(3) - 1)x + x^2 \right) \frac{1}{\sqrt{\lambda}} + \cdots \right]. \]

where we conveniently defined \( x = J - 2 \). The overall factor of \( x \) guarantees that the energy momentum tensor has protected dimension.

On the other hand, at strong coupling the Reggeon spin was computed using the dual string description\(^9, 4\)

\[ j(\nu) = 2 - \sum_{n=1}^{\infty} \frac{j_n(\nu^2)}{g^n} = 2 - \frac{4 + \nu^2}{2\sqrt{\lambda}} \left( 1 + \sum_{n=2}^{\infty} \frac{\tilde{j}_n(\nu^2)}{\lambda^{(n-1)/2}} \right), \]

where \( \tilde{j}_n(\nu^2) \), defined for \( n \geq 2 \), is a polynomial of degree \( n \). The \( n = 1 \) term in this expansion was computed in \(^9\) and gives the linear Regge trajectory of strings in the flat space limit. The general form that constrains the degree of the polynomial \( \tilde{j}_n(\nu^2) \) was derived in \(^4\) by requiring that such limit is well defined.
Noting that $-\Delta (\Delta - 4) = 4 + \nu^2$, we can equate both expansion (8) and (9) to obtain new data for the polynomials $\tilde{j}_n(\nu^2)$, with $n \geq 2$, that characterise this Regge trajectory. Writing

$$\tilde{j}_n(\nu^2) = \sum_{k=0}^{n-2} c_{n,k} \nu^{2k},$$

we can fix the coefficients $c_{n,n-2}$ and $c_{n,n-3}$. More precisely, we obtained that

$$c_{2,0} = \frac{1}{2}, \quad c_{3,1} = \frac{3}{8}, \quad c_{3,1} = \frac{3}{32}(8\zeta(3) - 7), \quad c_{5,2} = \frac{21}{64},$$

and the remaining coefficients of this type vanish ($c_{n,n-2} = 0$ for $n \geq 4$, $c_{n,n-3} = 0$ for $n \geq 6$). In particular, we derived the next and the next to next leading order correction to the intercept.

$$j(0) = 2 - \frac{2}{\sqrt{\lambda}} - \frac{1}{\lambda} + \frac{1}{4\lambda^{3/2}} + \ldots.$$  

Figure 3 shows this expansion in comparison with the weak coupling one. More recently, the corrections of order $\lambda^{-2}$ and of order $\lambda^{-5/2}$ and $\lambda^{-3}$ were respectively computed in\textsuperscript{10} and\textsuperscript{11}.

**Acknowledgments**

The research leading to these results has received funding from the [European Union] Seventh Framework Programme [FP7-People-2010-IRSES] under grant agreements No 269217 and 317089 and from the grant CERN/FP/123599/2011. Centro de Física do Porto is partially funded by the Foundation for Science and Technology of Portugal (FCT).

**References**