Lattice studies of pseudo-PDFs

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- Large international effort aiming at their measurement
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Lattice traditionally

- Calculation of Mellin moments of PDFs through matrix elements of twist−2 operators.
- Would not be an issue if every moment were accessible because a probability distribution is completely determined once all its moments are known.
- These studies are limited to the first few (three) moments due to
  - bad signal to noise ratio
  - power-divergent mixing on the lattice (discretized space-time does not possess the full rotational symmetry of the continuum).
Global PDF fits

- Realize a QCD analysis of hard-scattering measurements employing a variety of hadronic observables.
- Parton densities parametrized @ an initial energy scale evolved up to the scale of data via DGLAP eqs.
- Build theoretical predictions for the observables.
- Best fit parameters determined by the minimization of an appropriate figure of merit (eg. $\chi^2$).
- Many free parameters
- Advanced techniques (eg. use of neural networks).
Light-like is a NO-GO

Computing PDFs in LQCD we start from the equal time hadronic matrix element with the quark and anti-quark fields separated by a finite distance. For non-singlet parton densities the matrix element

\[ M^\alpha(z, p) \equiv \langle p | \bar{\psi}(0) \gamma^\alpha \hat{E}(0, z; A) \tau_3 \psi(z) | p \rangle \]

where \( \hat{E}(0, z; A) \) is the \( 0 \rightarrow z \) straight-line gauge link in the fundamental representation, \( \tau_3 \) is the flavor Pauli matrix, and \( \gamma^\alpha \) is a gamma matrix. We can decompose the matrix element due to Lorentz invariance as

\[ M^\alpha(z, p) = 2p^\alpha M_p(-(zp), -z^2) + z^\alpha M_z(-(zp), -z^2) \]
From the $\mathcal{M}_p(-(zp), -z^2)$ part the twist-2 contribution to PDFs can be obtained in the limit $z^2 \to 0$.

By taking $z = (0, 0, 0, z_3)$, $\alpha$ in the temporal direction i.e. $\alpha = 0$, and the hadron momentum $p = (p^0, 0, 0, p)$ the $z^\alpha$-part drops out.

The Lorentz invariant quantity $\nu = -(zp)$, is the "Ioffe time" (B. L. Ioffe, Phys. Lett. 30B, 123 (1969))

and \[ \langle p | \bar{\psi}(0) \gamma^0 \hat{E}(0, z; A)\tau_3 \psi(z) | p \rangle = 2p^0 \mathcal{M}_p(\nu, z_3^2) \]
Formalism

- the quasi-PDF $Q(x, p^2)$ is related to $M_p(\nu, z_3^2)$ by

$$Q(x, p^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \ e^{-ix\nu} \ M_p(\nu, [\nu/p]^2)$$

Quasi PDF mixes invariant scales until $p_z$ is effectively large enough

- while the pseudo-PDF

$$P(x, z_0^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \ e^{-i\nu\nu} \ M_p(\nu, z_0^2)$$

Pseudo PDF has fixed invariant scale dependence
loffe time PDFs $\mathcal{M}(\nu, z_3^2)$ defined at a scale $\mu^2 = 1/z_3^2$ are the Fourier transform of regular PDFs $f(x, \mu^2)$. (I.I. Balitsky and V.M. Braun, Nucl. Phys. B311, 541 (1988), V. Braun, et. al Phys. Rev. D 51, 6036 (1995))

$$\mathcal{M}(\nu, z_3^2) = \int_{-1}^{1} dx f(x, 1/z_3^2) e^{ix\nu}$$

Scale dependence of the loffe time PDF derived from the DGLAP evolution of the regular PDFs.

Ioffe time PDFs evolution equation

$$\frac{d}{d \ln z_3^2} \mathcal{M}(\nu, z_3^2) = -\frac{\alpha_s}{2\pi} C_F \int_0^1 du B(u) \mathcal{M}(u\nu, z_3^2)$$

with $B(u) = \left[\frac{1+u^2}{1-u}\right]_+$, $C_F = 4/3$, and $B(u)$ is the LO evolution kernel for the non-singlet quark PDF. (V. Braun, et. al Phys. Rev. D 51, 6036 (1995))
\[ z_3 \ll \rightarrow M_p(\nu, z_3^2) = M(\nu, z_3^2) + O(z_3^2) \]

But.... large \( O(z_3^2) \) corrections prohibit the extraction.

Conservation of the vector current implies \( M_p(0, z_3^2) = 1 + O(z_3^2) \).

but in a ratio \( z_3^2 \) corrections (related to the transverse structure of the hadron) might cancel (A. Radyushkin Phys.Lett. B767 (2017))

\[ M(\nu, z_3^2) \equiv \frac{M_p(\nu, z_3^2)}{M_p(0, z_3^2)} \]

- much smaller \( O(z_3^2) \) corrections and therefore this ratio could be used to extract the Ioffe time PDFs

- a well defined continuum limit and does not require renormalization
First case study in an unphysical setup

- Quenched approximation
- $32^3 \times 64$ lattices with $a = 0.093\text{fm}$.
- $m_\pi = 601\text{MeV}$ and $m_N = 1411\text{MeV}$

Now employing dynamical ensembles (preliminary)

<table>
<thead>
<tr>
<th>$a$(fm)</th>
<th>$M_\pi$(MeV)</th>
<th>$\beta$</th>
<th>$L^3 \times T$</th>
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<tbody>
<tr>
<td>0.127(2)</td>
<td>440</td>
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<td>0.094(1)</td>
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Table: Parameters for the lattices generated by the JLab/W&M collaboration using 2+1 flavors of clover Wilson fermions and a tree-level tadpole-improved Symanzik gauge action. The lattice spacings, $a$, are estimated using the Wilson flow scale $w_0$. Stout smearing implemented in the fermion action makes the tadpole corrected tree-level clover coefficient $c_{SW}$ used, to be very close to the value determined non-pertubatively with the Schrödinger functional method.
Following, C. Bouchard et al. Phys. Rev. D 96, no. 1, 014504 (2017) we compute a regular nucleon two point function

\[ C_p(t) = \langle N_p(t)\overline{N}_p(0) \rangle, \]

and

\[ C_p^{O^0(z)}(t) = \sum_{\tau} \langle N_p(t)O^0(z, \tau)\overline{N}_p(0) \rangle \]

with

\[ O^0(z, t) = \overline{\psi}(0, t)\gamma^0\tau_3\hat{E}(0, z; A)\psi(z, t) \]

Proton momentum and displacement of the quark fields along the \( \hat{z} \) axis

\[ M_{\text{eff}}(z_3p, \frac{z_2}{2p^0}; t) = \frac{C_p^{O^0(z)}(t + 1)}{C_p(t + 1)} - \frac{C_p^{O^0(z)}(t)}{C_p(t)} \]

Extract the desired m. e. \( J \) at large Euclidean time separation as

\[ \frac{J(z_3p, \frac{z_2}{2p^0})}{2p^0} = \lim_{t \to \infty} M_{\text{eff}}(z_3p, \frac{z_2}{2}; t) \]

where \( p^0 \) is the energy of the nucleon.
Numerical implementation

- Renormalization of the m.e.?

- For $z_3 = 0$ $\mathcal{M}(z_3 p, z_3^2) \rightarrow$ the local iso-vector current, should be $= 1$ (but ...) lattice artifacts...

- Introduce an RC $Z_p = \frac{1}{\mathcal{J}(z_3 p, z_3^2)|_{z_3=0}}$

- $Z_p$ has to be independent from $p$. But lattice artifacts or potential fitting systematics ...

- renormalize the m. e. for each momentum with its own $Z_p \rightarrow$ maximal statistical correlations to reduce statistical errors, and cancellation of lattice artifacts in the ratio
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- in practise use the double ratio

\[
M(\nu, z_3^2) = \lim_{t \to \infty} \frac{M_{\text{eff}}(z_3 p, z_3^2; t)}{M_{\text{eff}}(z_3 p, z_3^2; t)|_{z_3=0}} \times \frac{M_{\text{eff}}(z_3 p, z_3^2; t)|_{p=0, z_3=0}}{M_{\text{eff}}(z_3 p, z_3^2; t)|_{p=0}},
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- infinite \( t \) limit is obtained with a fit to a constant for a suitable choice of a fitting range.
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Nucleon dispersion relation. Energies and momenta are in lattice units. The solid line is the continuum dispersion relation (not a fit) while the errorband is an indication of the statistical error of the lattice nucleon energies.
Typical fits used to extract the reduced matrix element (here \( p = 2\pi/L \cdot 2 \) and \( z = 4 \) (LHS) and \( p = 2\pi/L \cdot 3 \) and \( z = 8 \) (RHS)). The average \( \chi^2 \) per degree of freedom was \( \mathcal{O}(1) \). All fits are performed with the full covariance matrix and the error bars are determined with the jackknife method.
Re and Im parts of $M(\nu, z_3^2)$. Curves plotted for comparison, given by Re and Im Fourier trasfos of $q_v(x) = \frac{315}{32} \sqrt{x}(1 - x)^3$. The data are approximately described by the same curve. This phenomenon can be understood if an approximate factorization of the longitudinal and transverse structure of the hadron occurs.
Residual $z_3$-dependence

- Data plotted as a function of the Ioffe time we can see that there is a residual $z_3$-dependence.

- This is more visible when, for a particular $\nu \rightarrow$ several data points corresponding to different values of $z_3$.

- Different values of $z_3^2$ for the same $\nu$ correspond to the loffe time distribution at different scales.
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- Is the residual scatter in the data points consistent with evolution? By solving the evolution equation at LO, the Ioffe time PDF at $z'_3$ is related to the one at $z_3$ by

$$M(\nu, z'^2_3) = M(\nu, z^2_3) - \frac{2}{3} \frac{\alpha_s}{\pi} \ln\left(\frac{z'^2_3}{z^2_3}\right) \int_0^1 du B(u) M(u\nu, z^2_3)$$

- only applicable at small $z_3$

- Check its effect using data at values of $z_3 \leq 4a$ corresponding to energy scales larger than 500 MeV.

- We fix the point $z'_3$ at the value $z_0 = 2a$ corresponding, at leading logarithm level, to the $\overline{MS}$-scheme scale $\mu_0 = 1$ GeV and evolve the rest of the points to that scale.
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Before and after evolution

The ratio $\mathcal{M}(\nu, z_3^2)$ for $z_3/a = 1, 2, 3, \text{ and } 4$. **LHS:** Data before evolution. **RHS:** Data after evolution. The reduction in scatter indicates that evolution collapses all data to the same universal curve.
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More on evolution

- LO evolution cannot be extended to very low scales.
- It is known that evolution stops below a certain scale (by observing our data we infer that this is the case for $z_3 \geq 6a$.)
- Adopt an evolution that leaves the PDF unchanged for length scales above $z_3 = 6a$ and use the leading perturbative evolution formula to evolve to smaller $z_3$ scales.
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Comparison to global fits

LHS: Data points for $\text{Re} \tilde{M}(\nu, z_0^2)$ with $z_3 \leq 10a$ evolved to $z_3 = 2a$. By fitting these evolved points with a cosine FT of $q_\nu(x) = N(a, b)x^a(1-x)^b$ we obtain $a = 0.36(6)$ and $b = 3.95(22)$ (statistical errors). RHS: Curve for $u_\nu(x) - d_\nu(x)$ built from the evolved data shown in the left panel and treated as corresponding to the $\mu^2 = 1 \text{ GeV}^2$ scale; then evolved to the reference point $\mu^2 = 4 \text{ GeV}^2$ of the global fits. 1-loop matching to $\overline{\text{MS}}$ still to be done on our data.

A. Radyushkin 1710.08813, Zhang et al 1801.03023, Izubuchi et al 1801.03917
Conclusions and outlook

- We presented a new approach for obtaining PDFs from lattice QCD calculations.
- Using an appropriate ratio of matrix elements we were able to get rid of UV divergences ensuring a well defined continuum limit.
- One can scan in Ioffe time $\nu$ which is the Fourier dual to the momentum fraction $x$ by using the hadron momentum.
- Large hadron momentum required to access the large $\nu$-regime or equivalently small-$x$ physics.
- To approach the light cone we need to send $z_3^2 \rightarrow 0$ keeping $\nu$ fixed.
Conclusions and outlook

- The pseudo-PDF ratio lead to suppression of scaling violations in $z_3^2$.

- The logarithmic singularity $(\ln(-z_3^2))$ of $\mathcal{M}(\nu, z_3^2)$ lead to DGLAP evolution.

- The observed $z^2$ dependence is compatible with DGLAP evolution.

- Soon we will be finalizing our results with $2 + 1$ dynamical flavors of Wilson clover fermions which will include a more detailed study of all involved systematics (discretization effects, finite-volume effects, lighter pions etc).
Stay Tuned!

for upcoming results . . .

Thanks a lot for your attention!!!