Caustic free completion of $P(X)$-fluids

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Possible pathologies in field theories

- Ghost instabilities. E.g., due to higher derivatives.
- Gradient instabilities.
Possible pathologies in field theories

- Ghost instabilities. E.g., due to higher derivatives.
- Gradient instabilities.
- Formation of caustics.


However, the underlying field equations do not exhibit any pathologies!

Not always the case: $P(X)$-fluids, k-essence, Generalized Galileons

Babichev’16.
Simple model

\[
S = \int d^4x \sqrt{-g} \frac{\chi^2}{2} (X - M^2) \implies T_{\mu\nu} = \chi^2 \partial_\mu \varphi \partial_\nu \varphi
\]

\[
X = (\partial_\mu \varphi)^2 \quad \text{Constraint} \quad X = M^2
\]

Projectable Horava–Lifshitz gravity or Mimetic matter scenario

Blas et al’09  Lim et al’10  Mukhanov and Chamseddine’13  Golovnev’13
Simple model

\[ S = \int d^4x \sqrt{-g} \frac{\lambda^2}{2} (X - M^2) \implies T_{\mu\nu} = \lambda^2 \partial_\mu \varphi \partial_\nu \varphi \]

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\[ T_{\mu\nu} = \rho u_\mu u_\nu \quad \rho = M^2 \chi^2 \quad u_\mu = \frac{\partial_\mu \varphi}{M}. \]

We deal with the irrotational pressureless perfect fluid.

It develops caustic singularities at some point!
Formation of caustics in pressureless perfect fluid

\[ X = (\partial_\mu \varphi)^2 = M^2 \implies \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = 0 \quad \mathbf{v} = -\frac{\nabla \varphi}{M} \]  

Euler equation

Formation of caustics = characteristics of equation of motion cross.

Effectively characteristics are trajectories of non-interacting dust particles. Along each trajectory the velocity is conserved \( \frac{dx}{dt} = v = \text{const.} \).
From mimetic matter to $P(X)$-fluid

\[
\int d^4x \sqrt{-g} \left[ \frac{\lambda^2}{2} X - \frac{M^2}{2} \lambda^2 \right] \implies \int d^4x \sqrt{-g} \left[ \frac{\lambda^2}{2} X - V(\lambda) \right].
\]

The field \( \lambda \) is not anymore the Lagrange multiplier, but the auxiliary field.

\[
\frac{V'(\lambda)}{\lambda} = X \implies S = \int d^4x \sqrt{-g} P(X).
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From mimetic matter to $P(X)$-fluid

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\int d^4 x \sqrt{-g} \left[ \frac{\lambda^2}{2} X - \frac{M^2}{2} \lambda^2 \right] \Rightarrow \int d^4 x \sqrt{-g} \left[ \frac{\lambda^2}{2} X - V(\lambda) \right].
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The field $\lambda$ is not anymore the Lagrange multiplier, but the auxiliary field.

\[
\frac{V'(\lambda)}{\lambda} = X \Rightarrow S = \int d^4 x \sqrt{-g} P(X).
\]

For example,

\[
V(\lambda) = -\frac{M^2 \lambda^2}{2} + \frac{M^4 \lambda^4}{4 \Lambda^4} \Rightarrow P(X) = X + \frac{X^2}{2} \quad \text{sub-luminal}.
\]

\[
V(\lambda) = \frac{M^2 \lambda^2}{2} - \frac{M^4 \lambda^4}{4 \Lambda^4} \Rightarrow P(X) = X - \frac{X^2}{2} \quad \text{super-luminal}.
\]
In the case of pressureless perfect fluid, along each characteristics $\frac{dx}{dt} = v$.

Generalization to $P(X)$-fluid:

$$\left( \frac{dx}{dt} \right)_\pm = \frac{v \pm c_s}{1 \pm vc_s}$$

$$v \equiv -\frac{\partial \varphi}{\partial t} \left[ \frac{\partial \varphi}{\partial x} \right]^{-1}$$

NB: the sound speed squared $c_s^2 = \left(1 + 2X \frac{P_{xx}}{P_x} \right)^{-1}$
In the case of pressureless perfect fluid, along each characteristics \( \frac{dx}{dt} = v \).

Generalization to \( P(X) \)-fluid:
\[
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\]

NB: the sound speed squared \( c_s^2 = \left( 1 + 2X \frac{P_{xx}}{P_x} \right)^{-1} \)

Specific cases: \( P(X) = X \implies c_s = 1 \implies \left( \frac{dx}{dt} \right)_\pm = \pm 1 \implies \text{No caustics.} \)

\( v = \pm 1 \implies \left( \frac{dx}{dt} \right)_\pm = \pm 1 \implies \text{No caustics} \quad \frac{\partial \varphi}{\partial t} = \mp \frac{\partial \varphi}{\partial x} \implies X = 0 \)
\[ \tau = \frac{\partial \varphi}{\partial t} \quad \chi = \frac{\partial \varphi}{\partial x} \]

Generic simple wave \( \tau = \tau(\chi) \)
The argument can be generalized to \textit{k}-essence (\( P(X, \varphi) \)-fluids).

Higher derivative terms in the \textit{Generalized Galileon} models do not cure caustic singularities.

First step towards curing caustic singularities: focus on \( P(X) \)-fluids.
Idea: promote $\lambda$ to a dynamical degree of freedom.

$$S = \int d^4x \sqrt{-g} \left( \frac{(\partial_\mu \lambda)^2}{2} + \frac{\lambda^2}{2} X - V(\lambda) \right).$$

$$\Psi = \lambda e^{i\varphi} \quad S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} |\partial_\mu \Psi|^2 - V(|\Psi|) \right].$$

Cf. Bilic’08  Bekenstein’88

The model is manifestly caustic free and exhibits luminal propagation.
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Cf. Bilic’08  Bekenstein’88

The model is manifestly caustic free and exhibits luminal propagation.

$$
V(|\Psi|) = V(\lambda) = \frac{M^2 \lambda^2}{2} \implies \text{mimetic matter}
$$

$$
V(\lambda) = -\frac{M^2 \lambda^2}{2} + \frac{M^4 \lambda^4}{4\Lambda^4} \implies P(X) = X + \frac{X^2}{2} \quad \text{sub-luminal}
$$

$$
V(\lambda) = +\frac{M^2 \lambda^2}{2} - \frac{M^4 \lambda^4}{4\Lambda^4} \implies P(X) = X - \frac{X^2}{2} \quad \text{super-luminal}
$$
Introducing a new degree of freedom may be not harmless

Before caustics are formed \((\partial_\mu \lambda)^2 \ll V(\lambda) \implies\)

Special initial conditions for \(\lambda\) are required!

At the time of the caustics formation \((\partial_\mu \lambda)^2 \sim V(\lambda)\)
\[ V(\lambda) = \frac{M^2 \lambda^2}{2} \]  
(mimetic matter, or pressureless fluid) \[ \implies \]

\[ \Psi = Ae^{iMt} + Be^{-iMt} \]

\[ B = 0 \implies \Psi = Ae^{iMt} \implies \lambda = |A| = \text{const} \]
\[ V(\lambda) = \frac{M^2 \lambda^2}{2} \quad \text{(mimetic matter, or pressureless fluid)} \quad \Rightarrow \]

\[ \Psi = Ae^{iMt} + Be^{-iMt} \quad B = 0 \quad \Rightarrow \quad \Psi = Ae^{iMt} \quad \Rightarrow \quad \lambda = |A| = \text{const} \]

\[ \Psi = \int d\mathbf{k} \alpha(\mathbf{k}) e^{i\mathbf{kx} + i\sqrt{k^2 + M^2}t} + \int d\mathbf{k} \beta(\mathbf{k}) e^{-i\mathbf{kx} - i\sqrt{k^2 + M^2}t} \]

\[ B = 0 \quad \Rightarrow \quad \beta(\mathbf{k}) = 0 \quad \Rightarrow \quad \Psi = \int d\mathbf{k} \alpha(\mathbf{k}) e^{i\mathbf{kx} + i\sqrt{k^2 + M^2}t} \]
Limit $k^2 \ll M^2$: connection to Schroedinger equation

$$\psi = e^{iMt} \int d\mathbf{k} \alpha(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x} + i\frac{k^2}{2M} t}$$

$$\psi = e^{iMt} \tilde{\psi} \implies i \frac{\partial \tilde{\psi}}{\partial t} = -\frac{\Delta \tilde{\psi}}{2M}$$
Limit $k^2 \ll M^2$: connection to Schroedinger equation

\[ \Psi = e^{iMt} \int dk \alpha(k) e^{ikx + i \frac{k^2}{2M} t} \quad \Psi = e^{iMt} \tilde{\Psi} \implies i \frac{\partial \tilde{\Psi}}{\partial t} = -\frac{\Delta \tilde{\Psi}}{2M} \]

Connection to pressureless perfect fluid

\[ \frac{\partial \rho}{\partial t} + \nabla (\rho v) = 0 \quad \rho = M^2 \lambda^2 \quad v = -\frac{\nabla \varphi}{M} \]

\[ \frac{\partial v}{\partial t} + (v \nabla)v = \frac{1}{2M^2} \nabla \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \]

Alternative to N-body simulations: Widrow and Kaiser’93, Uhlemann et al’14, Kopp et al’17 See the poster of Michael Kopp.

Physics of axions: Hui et al’17
Resolving caustic singularity

\[ \lambda(t = 0) \propto \exp\left(-\frac{x^2}{2L^2}\right) \]

\[ v(t = 0) \propto -x \implies v = -\frac{x}{T^s - t}. \]

\[ T^s \rightarrow T^s - iT \]

\[ \tau \propto \frac{1}{M} \quad v = -\frac{x(T^s - t)}{(T^s - t)^2 + \tau^2}. \]
Proper initial conditions: coupling phase to inflaton

During inflation

\[ Q = \lambda^2 \phi \propto \frac{1}{a^3} \implies Q \to 0 \]
Proper initial conditions: coupling phase to inflaton

During inflation \[ Q = \lambda^2 \dot{\varphi} \propto 1/a^3 \implies Q \to 0 \]

Break U(1)-invariance \[ S_{int} = \int d^4x \sqrt{-g} \cdot \beta \cdot \varphi \cdot T_{infl} \implies \]

\[ \implies Q(t) \approx \frac{\beta T_{infl}(t)}{3H(t)}. \]
Proper initial conditions: coupling phase to inflaton

During inflation  \[ Q = \lambda^2 \dot{\varphi} \propto 1/a^3 \implies Q \to 0 \]

Break U(1)-invariance  \[ S_{int} = \int d^4x \sqrt{-g} \cdot \beta \cdot \varphi \cdot T_{infl} \implies \]

\[ \implies Q(t) \simeq \frac{\beta T_{infl}(t)}{3H(t)}. \]

\[ \ddot{\lambda} + 3H \dot{\lambda} + V'(\lambda) - \frac{Q^2}{\lambda^3} = 0. \]

The field \( \lambda \) lives in the effective potential \( V_{eff} = V(\lambda) + \frac{Q^2}{2\lambda^2} \).

E.g. if \( V(\lambda) = \frac{M^2\lambda^2}{2} \), then \( \lambda \to \sqrt{\frac{Q}{M}} \).
Conclusions

- $P(X)$-fluid and mimetic matter scenario belong to the same class of models with two scalar fields. Both develop caustic singularities.

- Simple caustic free completion by means of the complex scalar.

- Mechanism of resolving caustic singularities: the collapse time is promoted to the complex number in the complete picture.

- Only specific initial conditions allow to reproduce mimetic matter scenario or $P(X)$-fluid.

- Initial conditions are set by inflation.
Thanks for your attention!