On solving post-Newtonian accurate Kepler Equation

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Eccentric GW templates

- We work on providing accurate & efficient prescriptions for $h_{+,\times}(t)$ associated with compact binaries merging along eccentric orbits

Eccentric GW templates

- We work on providing accurate & efficient prescriptions for $h_{+,\times}(t)$ associated with compact binaries merging along eccentric orbits

$$h_{+,\times}(t) = -\frac{G\, m\, \eta}{c^2\, R'} x \sum_{p=0}^{\infty} \left[\mathcal{C}^p_{+,\times} \cos(pl) + \mathcal{S}^p_{+,\times} \sin(pl) \right]$$

$$\begin{aligned}
\mathcal{C}^1_+ &= s_i^2 \left(-e + \frac{e^3}{8} - \frac{e^5}{192} + \frac{e^7}{9216} \right) \\
&\quad + c_{2\beta}(1 + c_i^2) \left(-\frac{3e}{2} + \frac{2e^3}{3} - \frac{37e^5}{768} + \frac{11e^7}{7680} \right) \\
\mathcal{S}^1_+ &= s_{2\beta}(1 + c_i^2) \left(-\frac{3e}{2} + \frac{23e^3}{24} + \frac{19e^5}{256} + \frac{371e^7}{5120} \right)
\end{aligned}$$

Eccentric GW templates

$$h_{+,\times}(r, \dot{r}, \phi, \dot{\phi})$$

Invoke quasi-Keplerian solution to the conservative dynamics $\Downarrow$

$$(r, \dot{r}, \phi, \dot{\phi}) \Rightarrow \kappa_i(x, e_t, u(t), m, \eta)$$

Explicit expression for $u(t)$ by solving Kepler equation $\Downarrow$

$$h_{+,\times}\left(\kappa_i(x, e_t, u(t), m, \eta)\right) \Rightarrow h_{+,\times}(t) \quad \text{for conservative dynamics}$$

Impose GW emission induced variations in x and e_t $\Downarrow$

$$h_{+,\times}\left(\kappa_i(x(t), e_t(t), u(t), m, \eta)\right) \Rightarrow h_{+,\times}(t) \quad \text{with radiation reaction}$$

$$\Downarrow$$

$h_{+,\times}(t)$ for binaries inspiralling in PN-accurate eccentric orbits

Eccentric GW templates

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$\Downarrow$

$h_{+,\times}(t)$ for binaries inspiralling in PN-accurate eccentric orbits

Classical Kepler problem

Equation of motion

$$\frac{d^2\vec{r}}{dt^2} = -\frac{Gm}{r^2}\hat{r}$$

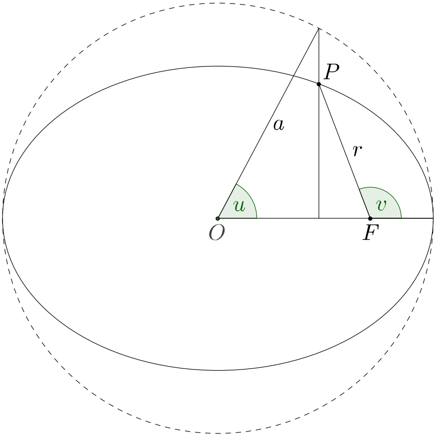
Classical Kepler problem

Keplerian parametrization

$$r = a\,(1 - e\cos u)$$
$$\phi - \phi_0 = v(u)$$

$$\tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}$$

where v is called true anomaly and u is called eccentric anomaly



Classical Kepler problem

Keplerian parametrization

$$r = a\,(1 - e\cos u)$$
$$\phi - \phi_0 = v(u)$$

Classical Kepler Equation

$$l = u - e\,\sin u$$

where $l = n\,(t - t_0)$ is called mean anomaly

Solving the classical Kepler Equation

Fourier series

$$u(l) - l = \sum_{s=1}^{\infty} A_s \sin(sl)$$

$$A_s = \frac{2}{\pi} \int_0^{\pi} (u(l) - l) \sin(sl) dl$$

Solving the classical Kepler Equation

Fourier series

$$u(l) - l = \sum_{s=1}^{\infty} A_s \sin(sl)$$

$$\begin{aligned}
A_s =& \frac{2}{\pi} \int_0^{\pi} (u(l) - l) \sin(sl) dl \\
=& \frac{2}{s\pi} \int_0^{\pi} \cos(sl) du \\
=& \frac{2}{s} \left\{ \frac{1}{\pi} \int_0^{\pi} \cos(su - se \sin u) du \right\} \\
=& \frac{2}{s} J_s(se)
\end{aligned}$$

Solving the classical Kepler Equation

Solution to the classical Kepler Equation

$$u = l + \sum_{s=1}^{\infty} \frac{2}{s} J_s(se) \sin(sl)$$

Relativistic Kepler problem

Equation of motion

$$\frac{d^2\vec{r}}{dt^2} = -\frac{Gm}{r^2}\hat{r} + A\,\hat{r} + B\,\hat{v}$$

$$A = A_1 + A_2 + A_{2.5} + A_3 + A_{3.5} + ...$$
$$B = B_1 + B_2 + B_{2.5} + B_3 + B_{3.5} + ...$$

- A's & B's are functions of r, $\dot{r}$, ϕ, $\dot{\phi}$, m and η
- Radiation reaction terms (2.5PN & 3.5PN) cause orbit to shrink
- There exists a quasi-Keplerian parametrization to the 3PN conservative dynamics

Relativistic Kepler problem

Quasi-Keplerian parametrization

$$
\begin{aligned}
r &= a_r\,(1 - e_r \cos u)\\
\phi - \phi_0 &= (1+k)v + (f_{4\phi} + f_{6\phi})\sin(2v) + (g_{4\phi} + g_{6\phi})\sin(3v)\\
&\quad + i_{6\phi}\sin(4v) + h_{6\phi}\sin(5v)\\
\tan\frac{v}{2} &= \sqrt{\frac{1+e_\phi}{1-e_\phi}}\tan\frac{u}{2}
\end{aligned}
$$

- The coefficients $a, e_r, e_\phi, k, f, g, i, h$ are PN-accurate functions of x, e_t and $\eta = \mu/m$
- k represents precession of periastron

Relativistic Kepler problem

3PN-accurate Kepler Equation

$$l = u - e_t \sin u + (g_{4t} + g_{6t})(v - u) + (f_{4t} + f_{6t}) \sin(v) + i_{6t} \sin(2v) + h_{6t} \sin(3v)$$

Relativistic Kepler problem

3PN-accurate Kepler Equation

$$
\begin{aligned}
l = {} & u - e_t \sin u + (g_{4t} + g_{6t})(v - u) \\
& + (f_{4t} + f_{6t}) \sin(v) + i_{6t} \sin(2v) + h_{6t} \sin(3v)
\end{aligned}
$$

Objective

Invert the 3PN-accurate Kepler Equation to get $u(l)$ accurate to 3PN order

Solving the 3PN-accurate Kepler Equation

$$v - u = 2 \sum_{j=1}^{\infty} \frac{\beta^j}{j} \sin(ju)$$

$$\sin(v) = \frac{2\sqrt{1-e_\phi^2}}{e_\phi} \sum_{j=1}^{\infty} \beta^j \sin(ju)$$

$$\sin(2v) = \frac{4\sqrt{1-e_\phi^2}}{e_\phi^2} \sum_{j=1}^{\infty} \beta^j \left(j\sqrt{1-e_\phi^2} - 1\right) \sin(ju)$$

$$\sin(3v) = \frac{2\sqrt{1-e_\phi^2}}{e_\phi^3} \sum_{j=1}^{\infty} \beta^j \left(4 - e_\phi^2 - 6j\sqrt{1-e_\phi^2} + 2j^2(1-e_\phi^2)\right) \sin(ju)$$

where $\beta = \frac{1-\sqrt{1-e_\phi^2}}{e_\phi}$

Solving the 3PN-accurate Kepler Equation

$$l = u - e_t \sin u + \sum_{j=1}^{\infty} \alpha_j(x, e_t, \eta) \sin(ju)$$

$$\begin{aligned}\alpha_j(x, e_t, \eta) =& 2\beta^j \frac{\sqrt{1-e_\phi^2}}{e_\phi^3} \left((f_{4t} + f_{6t}) e_\phi^2 + \frac{(g_{4t} + g_{6t}) e_\phi^3}{j\sqrt{1-e_\phi^2}} \right. \\ & + 2 i_{6t} e_\phi \left[j\sqrt{1-e_\phi^2} - 1 \right] \\ & \left. + h_{6t} \left[4 - e_\phi^2 - 6j\sqrt{1-e_\phi^2} + 2j^2(1-e_\phi^2) \right] \right)\end{aligned}$$

Solving the 3PN-accurate Kepler Equation

Fourier series

$$u - l = \sum_{s=1}^{\infty} A_s \sin(sl)$$

$$\begin{aligned} A_s &= \frac{2}{\pi} \int_0^{\pi} (u - l) \sin(sl) dl \\ &= \frac{2}{s\pi} \int_0^{\pi} \cos(sl) du \\ &= \frac{2}{s} \left\{ \frac{1}{\pi} \int_0^{\pi} \cos\left(su - se \sin u + s \sum_{j=1}^{\infty} \alpha_j \sin(ju) \right) du \right\} \end{aligned}$$

Solving the 3PN-accurate Kepler Equation

$$
\begin{aligned}
A_s &= \frac{2}{s\pi}\int_0^{\pi}\cos\left(su - se_t\sin u\right)du - \frac{2}{\pi}\sum_{j=1}^{\infty}\alpha_j\int_0^{\pi}\sin\left(su - se_t\sin u\right)\sin(ju)du \\
&= \frac{2}{s}J_s(se_t) + \frac{1}{\pi}\sum_{j=1}^{\infty}\alpha_j\int_0^{\pi}\Big\{\cos\left((s+j)u - se_t\sin u\right) - \cos\left((s-j)u - se_t\sin u\right)\Big\}du \\
&= \frac{2}{s}J_s(se_t) + \sum_{j=1}^{\infty}\alpha_j\left\{J_{s+j}(se_t) - J_{s-j}(se_t)\right\}
\end{aligned}
$$

Solving the 3PN-accurate Kepler Equation

Solution to the 3PN-accurate Kepler Equation

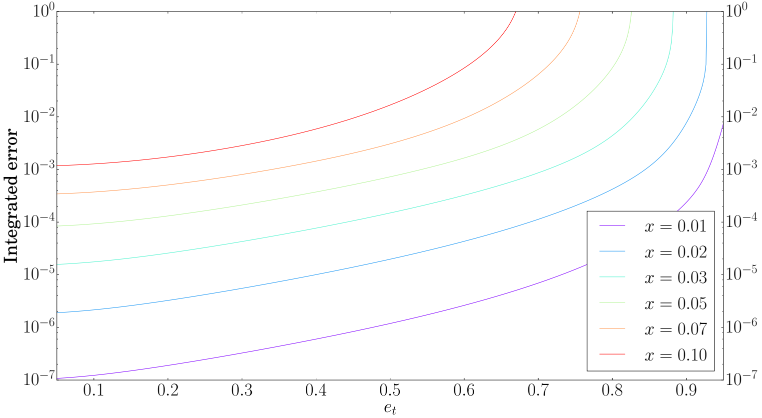
$$u = l + \sum_{s=1}^{\infty} A_s \sin(sl)$$

$$A_s = \frac{2}{s} J_s(se_t) + \sum_{j=1}^{\infty} \alpha_j \left\{ J_{s+j}(se_t) - J_{s-j}(se_t) \right\}$$

Numerical comparison

- Numerical solution using Newton's method
- We plot integrated error $\frac{1}{2\pi}\left(\int_0^{2\pi}\left(\frac{u_{\mathrm{num}}-u_{\mathrm{anl}}}{u_{\mathrm{num}}-l}\right)^2 dl\right)^{1/2}$

Numerical comparison



Brief summary

- We derived an elegant series solution to the 3PN accurate Kepler equation
- We use this solution to calculate eccentric GW templates