Deformation of $SL(2,\mathbb{R})$ observables on the Light Cone

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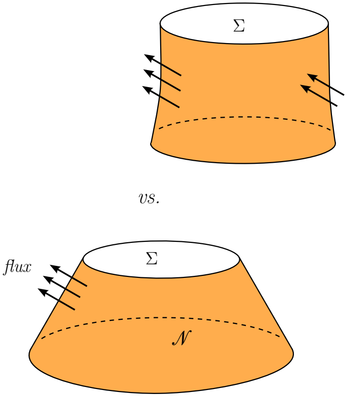
Introduction: subsystems as evolving regions in spacetime

To characterise a gravitational subsystem,
two choices must be made.

- A choice must be made for how to extend the boundary of the partial Cauchy hypersurface Σ into a worldtube $\mathscr{N}$.
- A choice must be made for what is the flux of gravitational radiation across the worldtube of the boundary, i.e. a (background field, c-number) that drives the time-dependence of the Hamiltonian.



N.B.: In spacetime dimensions $d < 4$, there are no gravitational waves, and we can forget about the second issue. The Hamiltonian will be automatically conserved.

To understand how gravity couples to boundaries, it is useful to work with differential forms rather than tensors since there is a natural notion of projection onto the boundary, namely the pull-back $\varphi^* : T^*M \to T^*(\partial M)$, which does not require a metric.

Tetrad defines the metric

$$g_{ab} = \eta_{\alpha\beta} e^{\alpha}{}_{a} e^{\beta}{}_{b}.$$

$\mathfrak{so}(1,3)$ connection and covariant derivative

$$\nabla_a V^{\alpha} = \partial_a V^{\alpha} + A^{\alpha}{}_{\beta a} V^{\beta}.$$

The commutator of two covariant derivatives defines the curvature,

$$[\nabla_a, \nabla_b] V^{\alpha} = F^{\alpha}{}_{\beta ab}\,[A] V^{\beta}.$$

There are two scalars that we can form out of the curvature tensor:

$$R[A,e] = F^{\alpha\beta}{}_{ab}[A] e_\alpha{}^a e_b{}^\beta,$$
$$R^*[A,e] = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta ab}[A] e_\mu{}^a e_\nu{}^b \approx 0.$$

Therefore, in the first-order formalism, there are *two coupling constants* at linear order in the curvature,

$$S = \frac{1}{16\pi G}\int_{\mathscr{M}} d^4v \Big[R - \frac{1}{\gamma}R^*\Big] + \text{boundary terms.}$$

G is Newton's constant, γ is the Immirzi parameter.

Using the isomorphism between spinors and tensors, the action splits into self-dual and anti-selfdual parts

$$S = \frac{1}{16\pi G}\int_{\mathscr{M}} d^4v\Big[R - \frac{1}{\gamma}R^*\Big] =$$
$$= \frac{\mathrm{i}}{8\pi\gamma G}(\gamma+\mathrm{i})\Big[\int_{\mathscr{M}} \Sigma_{AB}\wedge F^{AB}\Big] + \mathrm{cc.}$$

$SL(2,\mathbb{C})$ Spinor indices $A, B, C, \dots$ and $A', B', C', \dots$

Self-dual and anti-self-dual parts, e.g. of Plebański form

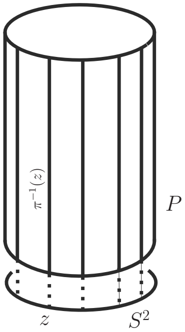
$$e_\alpha \wedge e_\beta =: \Sigma_{\alpha\beta} \equiv \Sigma_{AA'BB'} = -\bar{\epsilon}_{A'B'}\Sigma_{AB} - \epsilon_{A'B'}\bar{\Sigma}_{AB}$$

Field equations

$$\nabla \wedge \Sigma_{AB} = 0, \qquad F_{AB} = \Psi_{ABCD}\Sigma^{CD} = \Psi_{(ABCD)}\Sigma^{CD}.$$

Spacetime region bounded by null surface:

- Compact spacetime region $\mathscr{M}$.
- Bounded by spacelike disks M_0, M_1 and null surface $\mathscr{N}$.
- Null surface boundary $\mathscr{N}$ embedded into abstract bundle (ruled surface) $P(\pi, \mathscr{C}) \simeq \mathbb{R} \times \mathscr{C}$.
- Null generators $\pi^{-1}(z)$.



Metrical structures at the boundary:

- Signature $(0++)$ metric:
  $\varphi^*_{\mathcal{N}} g_{ab} = q_{ab} = 2m_{(a}\bar{m}_{b)}$.
- Null vectors: $l^a : q_{ab}l^b = 0 \Leftrightarrow \pi_* l^a = 0$.

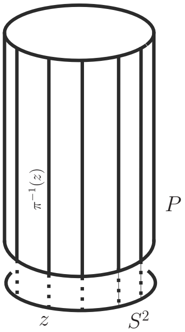
Abelian symmetries:

- $m_a \longrightarrow \mathrm{e}^{\mathrm{i}\varphi} m_a$
- $l^a \longrightarrow \mathrm{e}^{\lambda} l^a$

Associate spinors:

- Penrose null flag $\ell^A : l^a \simeq \mathrm{i}\,\ell^A \bar{\ell}^{A'}$
- Conjugate spinor-valued two-form
  $\eta_A \in \Omega^2(\mathcal{N} : \mathcal{S}_A)$.
- Area density $\varepsilon = \mathrm{i}\,\eta_A \ell^A \in \Omega^2(\mathcal{N} : \mathbb{R})$
- Abelian symmetries:

$$\begin{pmatrix} \ell^A \\ \eta_A \end{pmatrix} \longrightarrow \begin{pmatrix} \mathrm{e}^{+\frac{1}{2}(\lambda + \mathrm{i}\varphi)} \ell^A \\ \mathrm{e}^{-\frac{1}{2}(\lambda + \mathrm{i}\varphi)} \eta_A \end{pmatrix}$$



Bulk plus boundary action:

$$S = \frac{\mathrm{i}}{8\pi\gamma G}(\gamma + \mathrm{i})\left[\int_{\mathcal{M}} \Sigma_{AB} \wedge F^{AB} + \int_{\mathcal{N}} \eta_A \wedge \left(D - \tfrac{1}{2}\varkappa\right)\ell^A\right] + \mathrm{cc.}$$

Boundary conditions along $\mathcal{N}$: $\delta[\varkappa_a, l^a, m_a]/_\sim = 0$

- vertical diffeomorphisms $[\varphi^*\varkappa_a, l^a, \varphi^* m_a] \sim [\varkappa_a, \varphi_* l^a, m_a]$
- dilations $[\varkappa_a, l^a, m_a] \sim [\varkappa_a + \nabla_a f, \mathrm{e}^f l^a, m_a]$
- complexified conformal transformations $\lambda = \mu + \mathrm{i}\nu$: $[\varkappa_a, l^a, m_a] \sim \left[\varkappa_a - \frac{1}{\gamma}\nabla_a \nu, \mathrm{e}^\mu l^a, \mathrm{e}^{\mu + \mathrm{i}\nu} m_a\right]$
- shifts $[\varkappa_a, l^a, m_a] \sim [\varkappa_a + \bar{\zeta} m_a + \zeta \bar{m}_a, l^a, m_a]$

The equivalence class $g = [\varkappa_a, l^a, m_a]/_\sim$ characterises two degrees of freedom per point.

Covariant pre-symplectic potential for the partial Cauchy surfaces:

$$\Theta_\Sigma = \frac{\mathrm{i}}{8\pi\gamma G}(\gamma + \mathrm{i})\left[-\oint_{\mathscr{C}} \eta_A \mathrm{d}\ell^A + \int_\Sigma \Sigma_{AB} \wedge \mathrm{d}A^{AB}\right] + \mathrm{cc.}$$

Phase space of bulk and boundary degrees of freedom:

$$P_{phys} = (P_{bulk} \times P_{bndry})/\!\!/_{gauge}$$

Poisson brackets at the two-dimensional corner

$$\big\{\pi_A(z), \ell^B(z')\big\}_{\mathscr{C}} = \delta^B_A \delta^{(2)}(z, z').$$

Canonical (spinor-valued) momentum

$$\pi_A = \frac{\mathrm{i}}{8\pi G}\frac{\gamma + \mathrm{i}}{\gamma}\eta_A.$$

First results: Area operator and discrete spectra

- The cross-sectional oriented area is

$$\boxed{\mathrm{Area}[\mathscr{C}] = -8\pi G \frac{\mathrm{i}\gamma}{\gamma+\mathrm{i}} \oint_{\mathscr{C}} d^2x\, \pi_A \ell^A.}$$

- For the area to be real-valued (charge neutral), we have to satisfy the reality conditions,

$$\boxed{K - \gamma L = 0.}$$

- Generators of complexified $U(1)_{\mathbb{C}}$ transformations

$$L = -\frac{1}{2\mathrm{i}}\pi_A \ell^A + \mathrm{cc.} \quad \textit{(generator of U(1) transformations)},$$
$$K = -\frac{1}{2}\pi_A \ell^A + \mathrm{cc.} \quad \textit{(dilatations of the light like direction).}$$

- Boundary modes: creation and annihilation operators (half densities)

$$a^A = \frac{1}{\sqrt{2}}\Big[\sqrt{d^2\Omega}\,\delta^{AA'}\bar{\ell}_{A'} - \frac{\mathrm{i}}{\sqrt{d^2\Omega}}\pi^A\Big],$$
$$b^A = \frac{1}{\sqrt{2}}\Big[\sqrt{d^2\Omega}\,\ell^A + \frac{\mathrm{i}}{\sqrt{d^2\Omega}}\delta^{AA'}\bar{\pi}_{A'}\Big].$$

Fock quantisation of the area at the boundary

- Boundary Fock vacuum in the continuum

$$\forall z \in \mathscr{C} : a^A(z)\big|\{d^2\Omega, n_\alpha\}, 0\big\rangle = 0,$$
$$b^A(z)\big|\{d^2\Omega, n_\alpha\}, 0\big\rangle = 0.$$

- Boundary operators in terms of harmonic oscillators:

$$\hat{L}(z) = \frac{1}{2}\big[a^\dagger_A(z)a^A(z) - b^\dagger_A(z)b^A(z)\big],$$
$$\hat{K}(z) = \frac{1}{2\mathrm{i}}\big[a_A(z)b^A(z) - \text{hc.}\big],$$
$$\boxed{\big[\hat{K}(z) - \gamma\hat{L}(z)\big]\Psi_{\text{phys}} = 0.}$$

- $\hat{K}$ is a squeeze operator, $\hat{L}$ is difference of number operators.
- Area is quantised on physical states

$$\widehat{\text{Area}_\varepsilon}[\mathscr{C}]\Psi_{\text{phys}} = 4\pi\gamma\hbar G/c^3 \oint_{\mathscr{C}} \big[a^\dagger_A a^A - b^\dagger_A b^A\big]\Psi_{\text{phys}}.$$

$SL(2,\mathbb{R})$ variables and radiative modes

$SL(2,\mathbb{R})$ variables

Auxiliary two-dimensional vector space $\mathbb{V}$ with complex basis $(m^i, \bar{m}^i)$, $i = 0, 1$, and internal metric q_{ij}, q^{ij} : $q^{ik} q_{kj} = \delta^i_j$.

Fiducial dyad

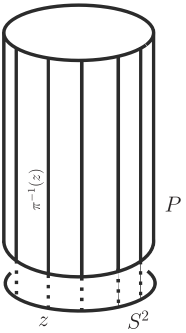
$$e^i_{(o)} = \bar{m}^i \frac{\mathrm{d}z}{1 + |z|^2} + \mathrm{cc.},$$
$$\delta[e^i_{(o)}] = 0.$$

Fiducial area

$$e^i_{(o)} \wedge e^j_{(o)} = \varepsilon^{ij} d^2 v_o.$$

Parametrisation of the dyad

$$\boxed{e^i = \Omega\, S^i{}_j\, e^j_{(o)}.}$$



Basic variables are now: $S^i{}_j : \mathcal{N} \to SL(2,\mathbb{R})$ and conformal factor $\Omega : \mathcal{N} \to \mathbb{R}$.

Second result: radiative modes do not see γ

Dirac bracket for $SL(2,\mathbb{R})$ radiative modes

$$\left\{ {S^i}_m(x), {S^j}_n(y) \right\}^* = -4\pi G\, \Theta(x,y)\, \delta^{(2)}(x,y)\, \Omega^{-1}(x)\, \Omega^{-1}(y) \times \left[\mathrm{e}^{-2\,\mathrm{i}\,(\Delta(x)-\Delta(y))} \left[X S(x) \right]^i{}_m \left[\bar{X} S(y) \right]^j{}_n + \mathrm{cc.} \right].$$

Dirac observables can be constructed using standard techniques.

Gauge symmetries:

1. $U(1)$ transformations
2. vertical diffeomorphisms along null generators

Summary

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- Action with Barbero–Immirzi parameter γ in causal regions
- γ mixes $U(1)$ frame rotations and dilations. **This is an important observation – it provides a simple geometric explanation for LQG quantum discreteness of geometry without relying on spin networks.**
- Poisson brackets for the boundary modes altered by addition of the Immirzi parameter.
- Poisson brackets for radiative modes unchanged by addition of the Immirzi parameter.

*ww, "Fock representation of gravitational boundary modes and the discreteness of the area spectrum", *Ann. Henri Poincaré* **18** (2017) 3695–3717, arXiv:1706.00479.
*ww, Gravitational $SL(2,\mathbb{R})$ algebra on the light cone, JHEP 07 (2021), 0957, arXiv:2104.05803.