THE BEST UNBIASED ESTIMATOR FOR THE CMB ANGULAR BISPECTRUM

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For those angular multipoles where cosmic variance is an issue, non-Gaussianities in the Cosmic Microwave Background (CMB) anisotropies will be hard to detect. Here, we construct explicitly the best unbiased estimator for the CMB angular bispectrum.

While the statistical properties of the CMB anisotropies are a powerful means to discriminate amongst the possible cosmological scenarios, actually measuring non-Gaussianity in the data is a very difficult task[1]. The typically small signal should be compared to the noise and the key quantity is the signal to noise ratio. The noise creeps into the dataset through instrumental errors, foregrounds contamination or incomplete sky coverage. Add to this the so-called 'cosmic variance': the fact that we only have access to one realization of the temperature anisotropies $\Delta(\mathbf{e}) \equiv \delta T/T(\mathbf{e})$ whereas theoretical predictions are expressed through ensemble averages. It can dominate the other sources of error and therefore, if one wants to unveil non-Gaussianity, it is necessary to address the cosmic variance problem for those quantities characterizing a possible non-Gaussian CMB temperature anisotropy distribution. For that one constructs estimators by performing spatial averages on the celestial sphere and finds the one which has the smallest possible variance. We here show the *best unbiased estimator* for the angular bispectrum $\mathcal{C}_{\ell_1\ell_2\ell_3}$ and we display the corresponding cosmic variance as well. Recall that $\mathcal{C}_{\ell_1\ell_2\ell_3}$ contains all the information available in the three-point correlation function, or its variants, like the skewness, collapsed and equilateral configurations[2]. The present analysis borrows from recent joint work with Jérôme Martin[3,4], to whom I am greatly *reconnaissant.*

Expanding the anisotropies over the microwave sky as usual

$$\Delta(\mathbf{e}) = \sum_{\ell m} a_\ell^m Y_\ell^m(\mathbf{e}) \tag{1}$$

the first three moments can be written as

$$\langle a_\ell^m \rangle = 0, \quad \langle a_{\ell_1}^{m_1} a_{\ell_2}^{m_2 *} \rangle = \mathcal{C}_{\ell_1} \delta_{\ell_1\ell_2} \delta_{m_1 m_2}, \quad \langle a_{\ell_1}^{m_1} a_{\ell_2}^{m_2} a_{\ell_3}^{m_3} \rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \mathcal{C}_{\ell_1\ell_2\ell_3}, \tag{2}$$

where $\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is a Wigner $3j$-symbol. The second equation ensures the isotropy of the CMB. The quantity $\langle a_{\ell_1}^{m_1} a_{\ell_2}^{m_2 *} \rangle$ is the second moment of the a_ℓ^m's and $\mathcal{C}_\ell$ is usually called the angular spectrum. In the third equation, the quantity $\langle a_{\ell_1}^{m_1} a_{\ell_2}^{m_2} a_{\ell_3}^{m_3} \rangle$ is the third moment while $\mathcal{C}_{\ell_1\ell_2\ell_3}$ is called the angular bispectrum. The presence of the $3j$-symbol guarantees that the third moment vanishes unless $m_1 + m_2 + m_3 = 0$ and $|\ell_i - \ell_j| \leq \ell_k \leq \ell_i + \ell_j$. Moreover, invariance under spatial inversions of the three-point function implies the additional rule [5,3] $\ell_1 + \ell_2 + \ell_3 = \text{even}$, in order for the third moment not to vanish. Finally, from this last relation and using standard properties of the $3j$-symbols, it follows that the angular bispectrum is left unchanged under any arbitrary permutation of the indices ℓ_i.

Let us call $\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3})$ the estimator for the angular bispectrum $\mathcal{C}_{\ell_1\ell_2\ell_3}$. The most general definition reads

$$\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3}) \equiv \int\int\int \mathrm{d}\Omega_1 \mathrm{d}\Omega_2 \mathrm{d}\Omega_3 E_{\mathrm{S}}^{\ell_1\ell_2\ell_3}(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)\Delta(\mathbf{e}_1)\Delta(\mathbf{e}_2)\Delta(\mathbf{e}_3). \tag{3}$$

where $E_{\mathrm{S}}^{\ell_1\ell_2\ell_3}$ is the weight function. The angular bispectrum is a real quantity and so is its estimator. Therefore, the weight function can be taken real. It is also symmetric under arbitrary permutations of directions $\mathbf{e}_i$. In addition, like $\mathcal{C}_{\ell_1\ell_2\ell_3}$, the weight function satisfies $E_{\mathrm{S}}^{\ell_1\ell_2\ell_3} = E_{\mathrm{S}}^{\ell_2\ell_1\ell_3}$, as well as for any other arbitrary permutation of the indices ℓ_i. The weight function can be expressed on the basis of the spherical harmonics as

$$E_{\mathrm{S}}^{\ell_1\ell_2\ell_3}(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3) = \sum_{\ell_1' m_1'}\sum_{\ell_2' m_2'}\sum_{\ell_3' m_3'} \mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\begin{smallmatrix}\ell_1' & \ell_2' & \ell_3'\\ m_1' & m_2' & m_3'\end{smallmatrix}\right\}} Y_{\ell_1'}^{m_1'}(\mathbf{e}_1) Y_{\ell_2'}^{m_2'}(\mathbf{e}_2) Y_{\ell_3'}^{m_3'}(\mathbf{e}_3). \tag{4}$$

The properties of the weight function imply that the coefficients d must satisfy

$$\mathrm{d}^{\ell_1\ \ell_2\ \ell_3 *}_{\left\{\begin{smallmatrix}\ell_1' & \ell_2' & \ell_3'\\ m_1' & m_2' & m_3'\end{smallmatrix}\right\}} = (-1)^{m_1'+m_2'+m_3'} \mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\begin{smallmatrix}\ell_1' & \ell_2' & \ell_3'\\ -m_1' & -m_2' & -m_3'\end{smallmatrix}\right\}} \quad , \quad \mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\begin{smallmatrix}\ell_1' & \ell_2' & \ell_3'\\ m_1' & m_2' & m_3'\end{smallmatrix}\right\}} = \mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\begin{smallmatrix}\ell_2' & \ell_1' & \ell_3'\\ m_2' & m_1' & m_3'\end{smallmatrix}\right\}}, \tag{5}$$

where the last relation is in fact valid for arbitrary permutations of any two columns of the collective subindex. Like the weight function, d is also left invariant under arbitrary permutations of indices ℓ_i (not primed). The estimator can be expressed in terms of the coefficients d and the a_ℓ^m's only: inserting the expansion of the weight function in the above expression for the estimator and using standard properties of the spherical harmonics one obtains

$$\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3}) = \sum_{\ell_1' m_1'}\sum_{\ell_2' m_2'}\sum_{\ell_3' m_3'} \mathrm{d}^{\ell_1\ \ell_2\ \ell_3 *}_{\left\{\begin{smallmatrix}\ell_1' & \ell_2' & \ell_3'\\ m_1' & m_2' & m_3'\end{smallmatrix}\right\}} a_{\ell_1'}^{m_1'} a_{\ell_2'}^{m_2'} a_{\ell_3'}^{m_3'}. \tag{6}$$

In practice, CMB observational settings are devised such that both the monopole and the dipole are subtracted from the anisotropy maps. This means that the coefficients d in the last equation are only non-vanishing for indices $\ell_i' \geq 2$ in the collective subindex. Moreover, the coefficients d satisfy $\ell_1 + \ell_2 + \ell_3 =$ even. We must now require that our general estimator given by Eq. (6) be unbiased, i.e. $\langle \mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3})\rangle = \mathcal{C}_{\ell_1\ell_2\ell_3}$. This forces the coefficients d to fulfill the following constraint

$$\sum_{m_1' m_2' m_3'} \mathrm{d}^{\ell_1\ \ell_2\ \ell_3 *}_{\left\{\begin{smallmatrix}\ell_1' & \ell_2' & \ell_3'\\ m_1' & m_2' & m_3'\end{smallmatrix}\right\}} \begin{pmatrix} \ell_1' & \ell_2' & \ell_3' \\ m_1' & m_2' & m_3' \end{pmatrix} = \delta_{\mathrm{S}}^{\ell_i \ell_j'}, \tag{7}$$

where we have defined a symmetrized Krönecker symbol for the ℓ multipole indices only, as follows $\delta_{\mathrm{S}}^{\ell_i\ell_j'} \equiv \frac{1}{6}(\delta_{\ell_1\ell_1'}\delta_{\ell_2\ell_2'}\delta_{\ell_3\ell_3'} +$ 5 additional permutations). It is easy to check that the constraint equation satisfies the conditions imposed by Eqns. (5) on the coefficients d. Using the previous properties for d, relabelling the indices $m_1' \leftrightarrow m_2'$ in Eq. (7) and finally noting that $\ell_1' + \ell_2' + \ell_3' = \ell_1 + \ell_2 + \ell_3 =$ even, which allows us to permute any two columns of the Wigner $3j$-symbol, one verifies that the left hand side of the constraint is invariant under $\ell_1' \leftrightarrow \ell_2'$. The same applies for any pair of ℓ multipole indices and this explains the presence of the symmetrized $\delta_{\mathrm{S}}^{\ell_i\ell_j'}$ in Eq. (7). We see from this that all coefficients d that do not satisfy $\ell_1' + \ell_2' + \ell_3' =$ even do not enter the constraint. These terms only increase the variance (which we want to minimize) and as a consequence one can take them equal to zero.

We are now in a position to calculate the variance of the estimator. Looking at Eq. (6) we see that this requires the computation of the sixth moment of the a_ℓ^m's. After having made use of the properties of the coefficients d and rearranging the resulting 15 terms into two groups, straightforward algebra yields

$$\begin{aligned}\langle[\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3})]^2\rangle &= \sum_{\ell_1' m_1'}\sum_{\ell_2' m_2'}\sum_{\ell_3' m_3'} \mathcal{C}_{\ell_1'}\mathcal{C}_{\ell_2'}\mathcal{C}_{\ell_3'} \\ &\times \left[6\mathrm{d}^{\ell_1\ \ell_2\ \ell_3 *}_{\left\{\begin{smallmatrix}\ell_1' & \ell_2' & \ell_3' \\ m_1' & m_2' & m_3'\end{smallmatrix}\right\}}\mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\begin{smallmatrix}\ell_1' & \ell_2' & \ell_3' \\ m_1' & m_2' & m_3'\end{smallmatrix}\right\}} + 9(-1)^{m_1'+m_2'}\mathrm{d}^{\ell_1\ \ell_2\ \ell_3 *}_{\left\{\begin{smallmatrix}\ell_1' & \ell_1' & \ell_3' \\ m_1' & -m_1' & m_3'\end{smallmatrix}\right\}}\mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\begin{smallmatrix}\ell_2' & \ell_2' & \ell_3' \\ m_2' & -m_2' & m_3'\end{smallmatrix}\right\}}\right].\end{aligned} \tag{8}$$

The square of the variance of $\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3})$ is given by $\sigma^2_{\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3})} = \langle[\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3})]^2\rangle - \langle\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3})\rangle^2$. Since departures from Gaussianity are expected to be small (specially on large angular scales), higher moments will be calculated in the mildly non-Gaussian approximation. Within this approximation we can write $a_\ell^m = a_\ell^{m(0)} + \epsilon\, a_\ell^{m(1)} + \mathcal{O}(\epsilon^2)$ where $a_\ell^{m(0)}$ is a Gaussian random variable and the expansion parameter ϵ is small. [The '(0)' label will be dropped out hereafter] Now, the term $\langle[\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3})]^2\rangle$ is of order ϵ^0 whereas the lowest non-vanishing order of $\langle\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3})\rangle^2$ is ϵ^2. Therefore, the latter one will not enter the minimization procedure and the variance squared will be written as $\sigma^2_{\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3})} \approx \langle[\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3})]^2\rangle$. This does not occur for the two-point correlator[6]; in that case both terms contributing to the square of the variance are of the same order in ϵ .

With a bit of effort one can see that the various contributions of the imaginary part of the coefficients d to the two terms, $6d^*d$ and $9d^*d$, only increase the variance. Since we know that a vanishing imaginary part does satisfy the constraint Eq. (7), it can be disregarded in the sequel. Therefore, Eq. (8) can then be written solely in terms of *real* coefficients d as follows

$$\sigma^2_{\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3})} = \sum_{\ell_1' m_1'}\sum_{\ell_2' m_2'}\sum_{\ell_3' m_3'} \mathcal{C}_{\ell_1'}\mathcal{C}_{\ell_2'}\mathcal{C}_{\ell_3'}\left[6\left(\mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\begin{smallmatrix}\ell_1' & \ell_2' & \ell_3' \\ m_1' & m_2' & m_3'\end{smallmatrix}\right\}}\right)^2 + 9(-1)^{m_1'+m_2'}\mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\begin{smallmatrix}\ell_1' & \ell_1' & \ell_3' \\ m_1' & -m_1' & m_3'\end{smallmatrix}\right\}}\mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\begin{smallmatrix}\ell_2' & \ell_2' & \ell_3' \\ m_2' & -m_2' & m_3'\end{smallmatrix}\right\}}\right].$$

Our *next move* now is to minimize this variance with respect to the coefficients d, taking into account the constraint of Eq. (7)

$$\delta\left\{\sigma^2_{\mathcal{E}(\mathcal{C}_{\ell_1\ell_2\ell_3})} + \sum_{\ell_1'\ell_2'\ell_3'}\lambda^{\ell_1\ell_2\ell_3}_{\ell_1'\ell_2'\ell_3'}\left[\sum_{m_1' m_2' m_3'}\mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\begin{smallmatrix}\ell_1' & \ell_2' & \ell_3' \\ m_1' & m_2' & m_3'\end{smallmatrix}\right\}}\begin{pmatrix}\ell_1' & \ell_2' & \ell_3' \\ m_1' & m_2' & m_3'\end{pmatrix} - \delta_{\mathrm{S}}^{\ell_i\ell_j'}\right]\right\} = 0. \tag{9}$$

Performing the variation δ having in mind that the symmetries of the coefficients d must be respected, we get

$$\begin{aligned}&12\mathcal{C}_{\ell_1'}\mathcal{C}_{\ell_2'}\mathcal{C}_{\ell_3'}\mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\begin{smallmatrix}\ell_1' & \ell_2' & \ell_3' \\ m_1' & m_2' & m_3'\end{smallmatrix}\right\}} + \lambda^{\ell_1\ell_2\ell_3}_{\ell_1'\ell_2'\ell_3'}\begin{pmatrix}\ell_1' & \ell_2' & \ell_3' \\ m_1' & m_2' & m_3'\end{pmatrix} \\ +\ &6(-1)^{m_2'}\mathcal{C}_{\ell_2'}\mathcal{C}_{\ell_3'}\delta_{\ell_1'\ell_2'}\delta_{m_1' -m_2'}\sum_{\ell m}\mathcal{C}_\ell(-1)^m\mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\begin{smallmatrix}\ell & \ell & \ell_3' \\ m & -m & m_3'\end{smallmatrix}\right\}} + \left[\begin{smallmatrix}1'\to 2' \\ 2'\to 3' \\ 3'\to 1'\end{smallmatrix}\right] + \left[\begin{smallmatrix}1'\to 3' \\ 2'\to 1' \\ 3'\to 2'\end{smallmatrix}\right] = 0.\end{aligned} \tag{10}$$

[...] terms are shorthand for the first one on the second line. This formula, together with Eq. (7), form a set of equations which completely determines the best unbiased estimator.

From this last equation and using the constraint Eq. (7) we can get the general expression for the Lagrange multipliers. Thus, we multiply Eq. (10) by the appropriate $3j$-symbol and we sum over the three indices m_i'. The first term is exactly the constraint and produces a $\delta_{\mathrm{S}}^{\ell_i\ell_j'}$. Using the fact that a triple sum over the m_i's of the squared of a $3j$-symbol gives unity, the second term yields the Lagrange multipliers themselves. Unfortunately, I don't have enough space to show that the last three terms vanish. Then, the Lagrange multipliers are given by

$$\lambda^{\ell_1\ell_2\ell_3}_{\ell_1'\ell_2'\ell_3'} = -12\mathcal{C}_{\ell_1'}\mathcal{C}_{\ell_2'}\mathcal{C}_{\ell_3'}\delta_{\mathrm{S}}^{\ell_i\ell_j'}. \tag{11}$$

Plugging this into Eq. (10), one has

$$12\mathcal{C}_{\ell_1'}\mathcal{C}_{\ell_2'}\mathcal{C}_{\ell_3'}\left[\mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\substack{\ell_1'\ \ell_2'\ \ell_3' \\ m_1'\ m_2'\ m_3'}\right\}} - \delta_{\mathrm{S}}^{\ell_i\ell_j'}\begin{pmatrix} \ell_1' & \ell_2' & \ell_3' \\ m_1' & m_2' & m_3' \end{pmatrix}\right] \tag{12}$$

$$+ \quad 6(-1)^{m_2'}\mathcal{C}_{\ell_2'}\mathcal{C}_{\ell_3'}\delta_{\ell_1'\ell_2'}\delta_{m_1'-m_2'}\sum_{\ell m}\mathcal{C}_\ell(-1)^m \mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\substack{\ell\ \ \ell\ \ \ell_3' \\ m\ -m\ m_3'}\right\}} + \left[\substack{1'\to 2' \\ 2'\to 3' \\ 3'\to 1'}\right] + \left[\substack{1'\to 3' \\ 2'\to 1' \\ 3'\to 2'}\right] = 0.$$

This is the final equation to be solved in order to determine the best unbiased estimator. A solution is

$$\mathrm{d}^{\ell_1\ \ell_2\ \ell_3}_{\left\{\substack{\ell_1'\ \ell_2'\ \ell_3' \\ m_1'\ m_2'\ m_3'}\right\}} = \begin{pmatrix} \ell_1' & \ell_2' & \ell_3' \\ m_1' & m_2' & m_3' \end{pmatrix}\delta_{\mathrm{S}}^{\ell_i\ell_j'}, \tag{13}$$

which leads to

$$\mathcal{E}_{\mathrm{Best}}(\mathcal{C}_{\ell_1\ell_2\ell_3}) = \sum_{m_1'm_2'm_3'}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1' & m_2' & m_3' \end{pmatrix} a_{\ell_1}^{m_1'}a_{\ell_2}^{m_2'}a_{\ell_3}^{m_3'}. \tag{14}$$

Seems familiar? An estimator restricted to the diagonal case $\ell_1 = \ell_2 = \ell_3$ (and then extended to $\ell_2 = \ell_1 + 2$ and $\ell_3 = \ell_1 - 2$) has been proposed [7,8] for $\mathcal{B}_\ell \equiv \mathcal{C}_{\ell\ell\ell}$. [a] The aim of these authors was not to seek the best estimator, but to use the corresponding, say, $\mathcal{E}(\mathcal{B}_\ell)$ to analyse the non-Gaussian features of the 4-yr COBE-DMR data (see also [9,10,11,12]). While their estimator is *not* unbiased, for it does not satisfy the constraint (7), by just removing an overall prefactor one gets our best unbiased estimator $\mathcal{E}_{\mathrm{Best}}(\mathcal{C}_{\ell_1\ell_2\ell_3})$, Eq. (14).

We now know the best unbiased estimator for $\mathcal{C}_{\ell_1\ell_2\ell_3}$ and then we can compute its variance, the smallest one amongst all possible estimator variances, which yields

$$\sigma^2_{\mathcal{E}_{\mathrm{Best}}(\mathcal{C}_{\ell_1\ell_2\ell_3})} = \mathcal{C}_{\ell_1}\mathcal{C}_{\ell_2}\mathcal{C}_{\ell_3}(1 + \delta_{\ell_1\ell_2} + \delta_{\ell_2\ell_3} + \delta_{\ell_3\ell_1} + 2\ \delta_{\ell_1\ell_2}\delta_{\ell_2\ell_3}). \tag{15}$$

We like to dub this (the square of) the 'bispectrum cosmic variance' in perfect analogy with $\sigma^2_{\mathcal{E}_{\mathrm{Best}}(\mathcal{C}_\ell)} = 2\mathcal{C}_\ell^2/(2\ell+1)$, which is (the square of) the variance of the best unbiased estimator for the angular spectrum, commonly known as the 'cosmic variance'.

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[a] $\mathcal{B}$ like $\mathcal{B}$ispectrum (like spe$\mathcal{C}$trum) ... or $\mathcal{C}$, $\mathcal{B}$, $\mathcal{A}$?, ... Hope the $\mathcal{T}$rispectrum will be called $\mathcal{T}$. Still, since no ambiguity arises I stick to $\mathcal{C}_{\ell_1\ell_2\ell_3}$ (for now).