WAVELET REGULARISED MAXIMUM ENTROPY

K.S. MAISINGER, M.P. HOBSON, R.B. BARREIRO
Astrophysics Group, Cavendish Laboratory, University of Cambridge,
Madingley Road, Cambridge CB3 0HE, England

Wavelet functions allow the sparse and efficient representation of a signal at different scales. Recently the application of wavelets to the denoising of maps of Cosmic Microwave Background (CMB) fluctuations has been proposed. We discuss a Maximum entropy method using wavelets for the denoising and deconvolution of CMB maps. A brief comparison of different filtering methods is given.

1 Introduction

Reconstructing maps of fluctuations in the Cosmic Microwave Background (CMB) radiation from observed data requires a number of steps in the data analysis. Among these are the removal of receiver noise, the deconvolution of the image and the separation and subtraction of foreground contaminants. The latter is usually achieved by the use of multifrequency observations, which exploit the different spectral behaviour of the physical foreground sources. Here we will consider the case of a single image and develop a maximum entropy method (MEM) for the reconstruction that involves a regularisation in wavelet space. In a simplified model we will assume that the data D observed by an experiment are given by a convolution of the true sky signal or image h with the point spread function P of the instrument, plus some Gaussian random noise ϵ:

$$D = P * h + \epsilon. \tag{1}$$

1.1 Wavelets

Wavelets are function basis sets that can be localised in both real and in Fourier space. The basis is constructed from dilations and translations of the original wavelet function. Thus wavelets can describe structure at a given scale and position on an image. There are many possible choices for the wavelet function; commonly known wavelets are the Haar and Daubechies-4 wavelets. The wavelet transform expands an image in terms of the wavelet basis functions. There exists a class of orthogonal wavelet transforms, analogous to Fourier transforms.

Two-dimensional transforms can be constructed from tensor products of the one-dimensional basis. However, it has been suggested that non-orthogonal, redundant, translationally and rotationally invariant (*isotropic*) transforms are better suited to the task of image reconstruction than orthogonal ones (Langer *et al.* [4]). A particular instance of a non-orthogonal transform is the à trous transform[6].

1.2 Maximum Entropy

In order to recover an image h from a blurred noisy version D, one needs to reconstruct a large number of parameters, the image pixels, from the data points. Clearly, some kind of regularisation is required to solve this inverse problem. A prescription for the reconstruction can be derived using Bayes' theorem. Bayes' theorem states that the probability of a reconstruction h given the data D is proportional to the likelihood of the data, times the prior probability of the reconstruction:

$$\mathrm{pr}(h|D) \propto \mathrm{pr}(D|h)\ \mathrm{pr}(h). \tag{2}$$

For Gaussian distributed errors the likelihood is given by $\mathrm{pr}(D|h) \propto \exp(-\chi^2)$, where χ^2 is the misfit statistics. The regularisation is achieved through the choice of a suitable prior. In the commonly used Maximum Entropy method[1], the prior is chosen to be $\mathrm{pr}(h) \propto \exp(\alpha S)$, where $S(h)$ is the (Shannon) entropy of the image, and the parameter α can be thought of as some Lagrange multiplier or equivalent thermodynamic temperature which controls the trade-off between the goodness-of-fit χ^2 and the regularisation. The reconstructed image can be calculated by minimising the functional $F = \chi^2 - \alpha S$.

2 Wavelet regularised MEM

The idea of wavelet regularised Maximum Entropy is to apply the entropy functional not in real space (i.e. on the image pixels), but on the wavelet coefficients of the image. This method was first proposed by Pantin and Starck[5]. However, these authors did not make use of the proper entropy functional for distributions that can take both positive and negative values (Hobson & Lasenby[3]):

$$S(h) = \Psi - 2m - h \log \frac{\Psi + h}{2m},$$

where $\Psi = \sqrt{h^2 + 4m^2}$ and the model m incorporates prior information about the image by setting some typical scale for the image *rms*, and is usually chosen to be constant across the image plane. Here we propose to set the the model m to the expected signal *rms* in the respective wavelet domain, $m = \sqrt{\sigma^2_{\mathrm{data}} - \sigma^2_{\mathrm{noise}}}$, and choose α proportional to the inverse of the model $\alpha \propto 1/m$. The absolute value of α can then be found by demanding that $\chi^2 = N$, the number of data points in the image. This choice is usually referred to as historic MEM.

3 Results

In order to test the method, we have applied it to simulated observations of CMB fluctuations taken from a CDM model. The fluctuations were simulated on a 256×256 grid with $1.5'$ resolution, corresponding to a total size of 6.4 degrees. The image was convolved with a 5 pixel FWHM beam ($7.5'$), and white Gaussian random pixel noise was added with varying signal-to-noise ratios S/N ranging from 5 to 1. The image was reconstructed with a wavelet regularised MEM using the orthogonal tensor transform and the à trous transform, a MEM algorithm in real space and in Fourier space, and with the orthogonal wavelet filter proposed by Sanz et al. [7]. The original and the reconstructed images are shown in Fig. 1. The Fourier and real space methods yield reconstructions that look visually similar to the original. However, they contain a large amount of small-scale structure, which is to a certain degree due to mere noise-fitting, whereas the wavelet filter reconstruction displays the characteristic horizontal and vertical stripes of the Daubechies-4 tensor wavelets used in the reconstruction. The wavelet regularised MEM reconstructions contain less small-scale structure.

In order to quantify the reconstruction errors, we have calculated the *rms* difference of original map and the reconstructions, quoted in per cent of the original image *rms* in Table 1. Not

S/N	5	2	1
wavelet regularised MEM			
à trous transform	11.5	18.2	25.9
orthogonal tensor transform	11.9	18.6	27.7
Fourier-space MEM	12.4	20.5	30.8
real-space MEM	13.0	20.9	33.2
wavelet filter (tensor wavelets)	17.2	21.9	28.6

Table 1: A comparison of reconstructions with signal-to-noise ratios S/N between 1 and 5. The numbers show the *rms* differences of original and reconstructed maps in per cent.

surprisingly, the quality of the reconstructions improves as the signal-to-noise ratio increases. Wavelet regularised MEM consistently performs best, with the errors on the à trous reconstructions lower than those using orthogonal wavelets. The Fourier method performs better than the real-space one. It should be noted that for Gaussian distributed images, such as CDM, Fourier MEM is to first order equivalent to Wiener filtering (Hobson *et al.*[2]). In the high signal-to-noise regime, the wavelet filter performs worst. However, it does not attempt a deconvolution, and its performance in the low signal-to-noise ratio, where the other methods lose their deconvolution capabilities, is quite good.

To conclude, wavelet regularised MEM entropy is the best Maximum Entropy algorithm we have.

Acknowledgments

We would like to thank Anthony Lasenby for useful discussions. KSM acknowledges support from an EU Marie Curie Fellowship. RBB acknowledges financial support from the PPARC in the form of a research grant.

References

1. S.F. Gull in *Maximum Entropy and Bayesian Methods*, ed. J. Skilling (Kluwer, Dordrecht, 1989).
2. M.P. Hobson *et al*, MNRAS **300**, 1 (1998)
3. M.P. Hobson and A.N. Lasenby, MNRAS **298**, 905 (1998).
4. W.D. Langer *et al.*, ApJ **408**, L45 (1993).
5. E. Pantin and J.-L. Starck, Astron. Astrophys. Suppl. **118**, 575 (1996).
6. J.-L. Starck *et al*, *Image Processing and Data Analysis*, (Cambridge University Press, 1998).
7. J.L. Sanz *et al*, preprint astro-ph/9906367

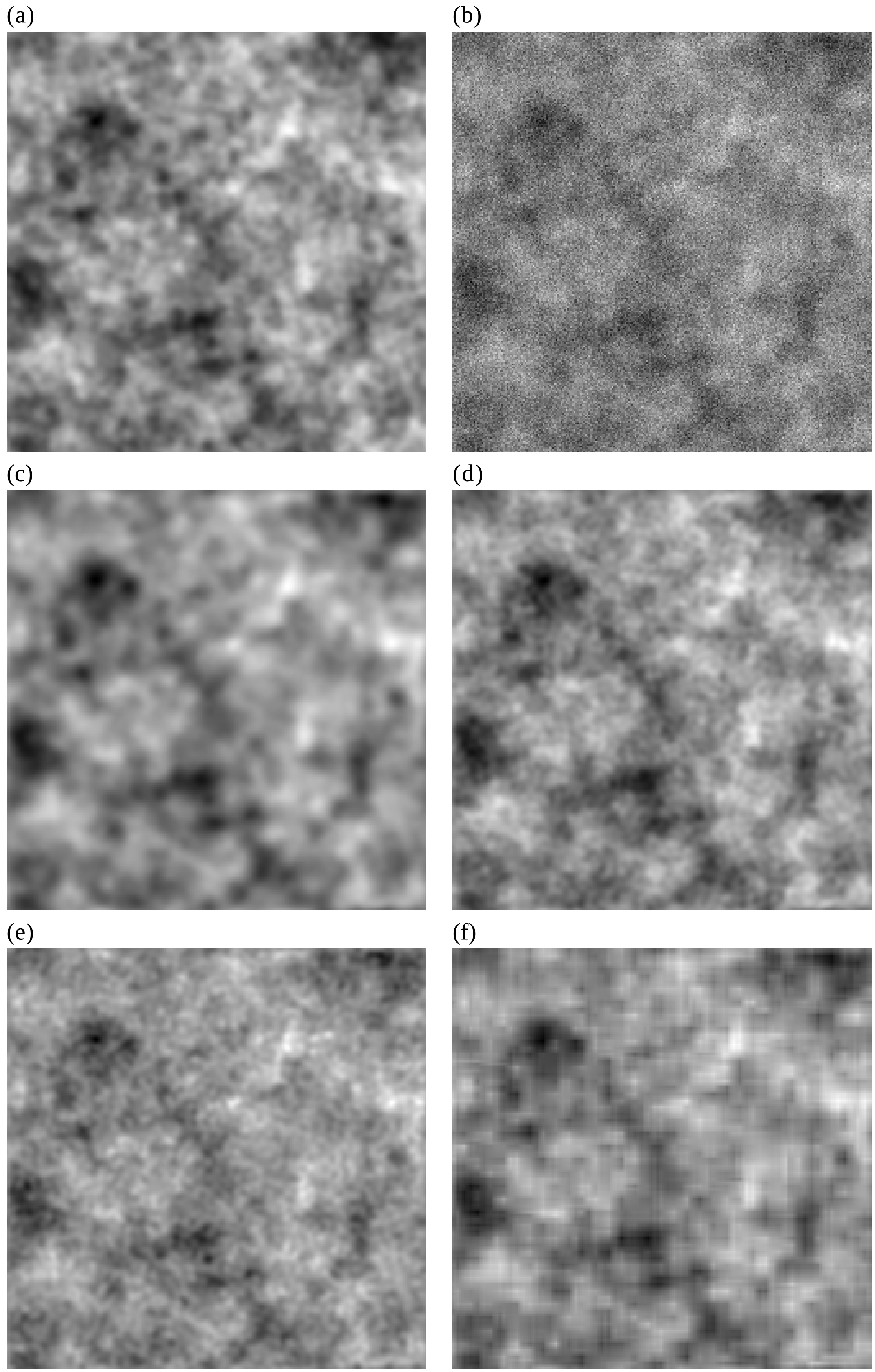

Figure 1: (a) Original CDM map. (b) Simulated data, convolved with a 5-pixel beam and noise added with a signal-to-noise ratio of $S/N = 1.$. (c) Reconstruction using à trous wavelet regularised MEM. (d) Fourier MEM. (e) Real-space MEM. (f) Tensor wavelet filter.