Exponential Potentials for Tracker Fields

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I. MOTIVATIONS

- The recent discovery of the acceleration of the Universe has produced an overwhelming number of papers on its possible explanations, alternative to a plain cosmological constant. The most favorite one is the introduction of a scalar field, usually minimally coupled with gravity. This poses the problem of the choice of a suitable potential.

- Moreover, as the equations are not exactly solvable in general, there is also a problem of qualitative analysis of the solutions and of the choice of initial conditions.

- Very crude approximations are made: "slow roll" condition, i.e.,

$$\dot{\varphi}^2 \ll V(\varphi)$$

.

- A complete treatment of general exact solutions of the Einstein equations in presence of both nonrelativistic matter and scalar field, with a suitable physical potential, and with a new treatment of the well known "tracking condition", $\Gamma > 1$ is possible.

- A result is that the slow roll condition is indeed not necessary and that insisting on it can bring to lose very interesting models.

- Closely connected to this, is a discussion on the meaning of the statement that the state equation $w \equiv p_\varphi / \rho_\varphi$ for the scalar field should be "almost constant". By means of an analytic example, we shall see that this expression is highly misleading, indeed.

- Our choice is

$$V(\varphi) = B^2 e^{-\sigma\varphi} \, , \tag{1}$$

with B arbitrary and $\sigma \equiv \sqrt{3/2}$ (in units $8\pi G = 1$).

- First of all, although exponential potential is "natural "in higher dimension theories, supergravity and superstring models, it is not generally considered as feasible.

- The usual objection to this potential is in fact that it cannot really be the right one, because "it scales approximately" like the matter component, so that, if the dark energy is dominant now, it should have been dominating also in the past, which is in contrast with BBN observations.

- This objection is wrong.

- A second common objection is that it describes a very peculiar situation, due to a fine tuning of initial conditions. It is possible, on the other hand, to find general and exact solutions.

- A third possible objection is that this model could be in contrast with the observations. As a matter of fact, some papers have already appeared, successfully comparing this model with present day observations.

- Our purpose is to show that this model we consider can emulate very well a standard model with dust plus a cosmological constant, well beyond any reasonable improvement of the observational techniques. In order to do this, a preliminary discussion on what is really measurable is again needed.

- A second result is that many of the current statements about tracking solutions are based on an incorrect result and should therefore be revisited.

II. HOW CAN WE SET THE CHOICE FOR THE POTENTIAL?

- There is no idea of which form the potential of the scalar field should take.

- The inverse power law, for instance, is certainly useful for a qualitative treatment of the problem, but only under the assumption of the slow rolling condition.

- The answer to this problem is simple in principle: the right form is the one that gives the best fit to the observed luminosity distance curve $d_L(z)$, and the fact that we cannot establish it firmly is only due to our limited observational precision.

- Indeed, it is possible to show that, whichever is the expression of $d_L(z)$, we can find, in principle, a suitable form for $V(\varphi)$, which gives it back. Unfortunately, the value of Ω_{0m} turns out to be *arbitrary*! The reason of this is due to the fact that the introduction of an arbitrary function in the model is equivalent to taking an infinite number of free parameters into account.

- In our opinion, the only possible way to escape from this *impasse* is to ask for potentials which have some possible explanation in terms of fundamental physics.

- The best candidates are the exponential potential, like that in Eq. (1) (but with σ undetermined), and polynomials of the form

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 + \lambda\varphi^4 + hig.\ ord. \tag{2}$$

III. WHAT IS REALLY MEASURABLE?

- We have a two-dimensional configuration space, $\{a, \varphi\}$ plus the *velocities* $\{\dot{a}, \dot{\varphi}\}$, the total number of variables is four. This is the maximum number of independent quantities which can be measured.

- Instead of these rather abstract objects, we can use derived quantities, e.g. Ω_{0m}. Its value can, in principle, be measured independently from the SNIa observations.

- q_0 is not directly measurable in practice. However, q_0 is also present in the second term of the series

$$d_L(z) = H_0^{-1}\left(z - \frac{(1-q_0)}{2}z^2 + ...\right) \ , \tag{3}$$

so that it seems possible to measure it.

- Eq. (3) after the second term is valid only at rather low redshifts (say $z \leq 0.1$). At these values, it is practically impossible to obtain sufficient precision, while at higher values the other terms cannot be neglected.

- What is thus left? What, for the moment firmly established, i.e., $h \approx 0.7$ and $\Omega_{0m} \approx 0.3$, with 3σ level $\sim 10\%$ and $\sim 50\%$ errors, respectively. These values are based on estimates not strictly dependent on the cosmological model.

- Also, a third useful variable could be the age of the universe t_0, for which we have only a rather poor estimate $t_0 > 13$ Gyr.

IV. WHICH IS THE PRECISE MEANING OF THE EXPRESSION “TRACKING BEHAVIOR”?

- Scalar fields allow to avoid the fine tuning of initial conditions, obtaining the same final behavior “for a wide range of initial possibilities”. (The sentence is ambiguous!). The situation is “similar but not exactly equal” to that of an attractor in the theory of dynamical systems.

- To be more precise. The equations which we have to consider are the Einstein equations plus the Klein-Gordon equation

$$3H^2 = \rho_\varphi + \rho_m \ , \tag{4}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{2}\dot{\varphi}^2 - V(\varphi) = 0 \ , \tag{5}$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0 \ . \tag{6}$$

- They can be derived by the following Lagrangian

$$L = 3a\dot{a}^2 - \frac{1}{2}a^3\dot{\varphi}^2 + a^3 V(\varphi) \, . \tag{7}$$

Eq. (4) is then obtained as conservation of the energy function $E_L \equiv (\partial L/\partial \dot{a})\, \dot{a} + (\partial L/\partial \dot{\varphi})\, \dot{\varphi} - L$.

- Liouville's theorem holds for this system. Thus, the phase space volume of possible initial conditions is conserved and has no attractors!!

- Nevertheless, the tracking behavior can be recovered by the following argument. Consider a set of independent variables for this system (not necessarily $\{a, \varphi, \dot{a}, \dot{\varphi}\}$) and a volume of possible initial conditions. During the evolution, the volume is deformed, and it is possible that its projection on a two(three)-dimensional subspace converges to a point or a line. It is clear that the result is strongly dependent on the choices of variables and projection. It is also clear that the gain in information on the variables of the subspace is compensated by a loss of information on the variables of the complement.

- In our case, only a part of the variables is practically measurable. Therefore we can make the usual choice for the subspace $\{\log a, \log \rho_\varphi\}$, and show that, for a large set of initial conditions of these two variables, the orbits converge to a line. Another possibility is to consider $\{H, \Omega_m\}$, and check that the orbits converge to a point.

- It is clear that only *observable* quantities should be taken into account. For any other parameter the concept of fine tuning is meaningless.

- The concept illustrated above is the only one which is relevant for the solution of the coincidence problem. However, In the literature the term "tracking" is often used in a different meaning. The idea is that a large set of initial conditions should converge towards a *nearly constant* w, well *before* the present epoch, and maintain this condition forever. This situation, although possible, is not at all necessary.

V. THE SOLUTION FOR THE EXPONENTIAL POTENTIAL CASE

By Noether symmetry (Capozziello et al. 1996), we choose the transformed variables

$$a^3 = uv \quad , \quad \varphi = -\frac{1}{\sigma} \log \frac{u}{v} \ . \tag{8}$$

Lagrangian (7) becomes

$$L = \frac{4}{3}\dot{u}\dot{v} + B^2 u^2 \ , \tag{9}$$

The related conserved quantity is

$$F \equiv \frac{\partial L}{\partial \dot{v}} = \frac{4}{3}\dot{u} = \frac{1}{3}(6\dot{a} + \sqrt{6}a\dot{\varphi})(a\exp(-\sigma\varphi))^{1/2} \tag{10}$$

the solution is

$$u = u_1 t + u_2 \ , \tag{11}$$

$$v = \frac{1}{6}u_1\omega t^3 + \frac{1}{2}u_2\omega t^2 + v_1 t + v_2 \ , \tag{12}$$

with $\omega \equiv \sigma^2 B^2$.

The relevant physical quantities are

$$H = \frac{(12\mathcal{H}_0 - 8)\, t^2 + 8 - 6\mathcal{H}_0}{(9\mathcal{H}_0 - 6)\, t^3 + (12 - 9\mathcal{H}_0)\, t} \ , \tag{13}$$

$$\Omega_m = \frac{(4 - 3\mathcal{H}_0)\left((3\ \mathcal{H}_0 - 2)\, t^2 - (3\mathcal{H}_0 - 4)\right)}{\left(2\,(3\mathcal{H}_0 - 2)\, t^2 - (3\mathcal{H}_0 - 4)\right)^2} \ , \tag{14}$$

$$w = -\frac{2\,(3\mathcal{H}_0 - 2)\, t^2 + 3\,(4 - 3\mathcal{H}_0)}{4\,(3\mathcal{H}_0 - 2)\, t^2 + 3\,(4 - 3\mathcal{H}_0)} \ . \tag{15}$$

$\mathcal{H}_0$is adimensional and it is not$h$. Here$H(t_0 = 1) \equiv \mathcal{H}_0$).

Then

$$\Omega_{0m} = -\frac{2\,(3\mathcal{H}_0 - 4)}{9\mathcal{H}_0^2} \quad , \quad w_0 = -\frac{8 - 3\mathcal{H}_0}{4 + 3\mathcal{H}_0} \ . \tag{16}$$

For $\mathcal{H}_0 = 1$, we have $\Omega_{0m} \approx 0.22$ and $w_0 \approx -0.71$. These values are reasonable for a best fit with SNIa data. Assuming $h = 0.7$, we get $t_0 \approx 14 \div 15\,Gyr$, which is also good. Moreover, we get

$$\frac{\dot{\varphi}_0^2}{2V(\varphi_0)} = \frac{1}{6} \ , \tag{17}$$

showing that the slow rolling condition is not fulfilled.

For a qualitative analysis, we can use

$$a^3 = \frac{t^2\left(t^2+1\right)}{2}\,, \quad H = \frac{4t^2+2}{3t^3+3t}\,, \quad \Omega_m = \frac{t^2+1}{\left(2t^2+1\right)^2}, \quad w = -\frac{2t^2+3}{4t^2+3}\,. \tag{18}$$

VI. COMPARISON WITH Λ-TERM MODEL AND OBSERVATIONS

- Our model is practically equivalent to a Λ-term model in accounting for present day observations.
- To do this we take advantage of the CAMB and CosmoMC codes which allow to compare a model with WMAP and CBI data, for the CMBR spectrum, and with SNIa data and other constraints coming from BBN and HST Key Project. The LSS power spectrum derived from the Two Degree Field (2dF) Galaxy Redshift Survey is also taken into account.

- The procedure consists in generating, from random sets of the parameters, thousands samples of results. The maximum likelihood is then obtained by product of the single likelihood functions as established by the observers teams (Details in Lewis & Bridle 2002,2003).

- The best fit results are reported in Table I. It is clear that the two sets of parameters are practically the same.

- In our model it is possible to get an analytic parametric expression for the luminosity distance in terms of hypergeometric and Euler functions:

$$z = \left(\frac{2}{t^2\,(t^2+1)}\right)^{1/3} - 1\,, \tag{19}$$

$$d_L(t, \mathcal{H}_0 = 1) = \frac{1}{(t^2+1)^{1/3}} \left(\frac{3\sqrt{\pi}\Gamma(7/6)}{\Gamma(2/3)t^{2/3}} - \frac{2^{2/3}\,{}_2F_1(1/6, 1/3, 7/6, -t^2)}{t}\right)\,. \tag{20}$$

- From this it is not difficult to compute the distance modulus and compare it with Λ-term model and SnIa data (Fig1). With the aid of CAMB code, we have also generated the CMBR spectrum for the two models (Fig.2). The overlapping of the plots is striking in both cases.

- It appears very unlikely that more precise observations, of the same type here considered, will disentangle this degeneracy. An independent precise measure of h and/or Ω_m could be, on the contrary, of great help.

VII. ANALYSIS OF W

- The notion of w *almost constant* is somewhat ambiguous: we have in fact to ask *almost constant with respect to which independent variable*? From Fig. 3 that the function $w\,(t)$ does not show any flatness except towards infinity, which can be also easily deduced from Eq. (15).

- The situation of $w\,(\log a)$, showed in Fig. 4. We have $w \sim -1$ in the past, with a sharp transition to the asymptotical value $w = -0.5$. The present time (i.e., $a_0 = 1$), marked by the vertical line, occurs just in the middle of this transition. For the range of observational data, we have to consider the epoch when w is mostly varying, that is, just the contrary of what is generally considered (Fig.5)!

- This situation looks like a cosmic coincidence, but it is not so *striking*. Indeed, Fig. 4 is done with the best fit values of Table I, but we can vary the present value of Ω_{0m} over a quite large range, much wider than any imaginable one, and obtain very similar results (Fig.6)

- The above example shows that, *in the context of our model*, the allowed value of the parameter forces to place the present epoch in the period of maximal variability of w, *with respect to* $\log a$. If we consider another dependence, and of course if we change the model, this is not necessarily true.

- This example indicates that the arguments involving variations of w are really very subtle, and that it may be dangerous to base our conclusions on rough approximations based on this type of arguments.

VIII. THE TRACKING BEHAVIOR

- Tracking behavior can be analyzed recasting the above solutions as

$$a^3 = \frac{1}{2}t^2(t^2+1) + \frac{1}{2}t(2-3t+t^3)\varepsilon\,, \tag{21}$$

$$\varphi = \sqrt{\frac{2}{3}}\log\left(\frac{t+t^3+(2-3t+t^3)\varepsilon}{2u_1^2 t}\right)\,, \tag{22}$$

$$H = \frac{2}{3t}\frac{t+2t^3+(1-3t+2t^3)\varepsilon}{t+t^3+(2-3t+t^3)\varepsilon}\,, \tag{23}$$

$$w = -\frac{2t^6+3t^4+(4t^6-6t^4+8t^3)\varepsilon+(2t^6-9t^4+8t^3-1)\varepsilon^2}{4t^6+3t^4+(8t^6-6t^4+4t^3)\varepsilon+(4t^6-9t^4+4t^3-1)\varepsilon^2}\,, \tag{24}$$

$$\Omega_m = \frac{t(t^3+t)-t(2t^3+6t-2)\varepsilon-t(3t^3-9t+6)\varepsilon^2}{4t^6+4t^4+t^2+(8t^6-8t^4+4t^3-6t^2+2t)\varepsilon+(4t^6-12t^4+4t^3+} \tag{25}$$

where ε is a combination of initial data which parametrizes the observable quantities, its values can range $0 \div 10^{-9}$ to match with observations!

- In Fig.7, the plot of w versus $\log_{10} a$ is shown. We see that the transition between $w \approx 1$ and $w \approx -1$ occurs at $z \approx 10^3$, as requested.

- In Fig. 8 (only the first transition is now shown).

- Now we pass to the more familiar plot of $\log_{10}\rho_\varphi$ versus $\log_{10} a$ (Fig. 9). Again, in the case 2), we can set the initial point between $z \approx 10^3$ and $z \approx 10^4$ and see that the values of $\log_{10}\rho_\varphi$ span the interval $(0, 0.6)$, so that ρ_φ can vary upon about 6 orders of magnitude. This is not the hundreds of them presented in the literature.

- In Fig. 10 we present the plot of H versus Ω_m. The situation seems very similar to the others, with a lot of initial conditions converging towards the tracker solution.

- The tracking behavior will be completely lost if we consider the variable φ, and leave undetermined its present value.

IX. THE TRACKING CONCEPT REVISITED

- A currently accepted point of view on tracking solutions is that the "attractor" should be characterized by a nearly constant value of w, well before the present epoch. Of course, the tracking solution must be reached before, but, from the above example, it clear that it does not imply w being constant, while, on the contrary, the present epoch occurs in the period when w changes most. The asymptotic constant value of w has not necessarily anything to do with observations.

- A second current point of view is that the potentials which are good for tracking should be such that

$$\Gamma \equiv \frac{VV''}{(V')^2} > 1\,, \tag{26}$$

with $\Gamma(\varphi)$ nearly constant over the possible range of φ. In this case, after some oscillations between -1 and 1, w should reach a stable nearly constant value, given by

$$w = \frac{w_B - 2(\Gamma - 1)}{1 + 2(\Gamma - 1)}\,, \tag{27}$$

where w_B is the state equation of the matter, in practice zero (Steinhardt et al. 1999).

- The only two potentials for which Γ is strictly constant are the exponential ($\Gamma = 1$) and the inverse power

$$V(\varphi) = M\varphi^{-\alpha}\,, \tag{28}$$

with $\alpha > 0$ ($\Gamma = (\alpha + 1)/\alpha$). In the first case, from Eq. (27) we get $w = w_B$, which seems to be untenable. In the second case, we have

$$w = \frac{\alpha w_B - 2}{\alpha + 2}\,. \tag{29}$$

These results have to be discussed and revised.

- In our example, the asymptotic value of w is $-1/2$, so that, also in the case of practically constant w, it does not scale as the background. In fact, it is not difficult to show that, with a generic exponential potential $V = A\exp(-\lambda\varphi)$, a scaling equal to the matter background is possible only for $\lambda > \sqrt{3}$. This seems to be in contradiction with Eq. (27), which makes no distinction on the value of λ.

- In the case of the double exponential

$$V(\varphi) = (Ae^{\sigma\varphi/2} - Be^{-\sigma\varphi/2})^2\,, \tag{30}$$

it is easy to show that it is $\Gamma < 1$ always, and yet it has proved to be suitable for tracking. This is not a contradiction with the tracking theorem, which only claims on sufficiency of $\Gamma > 1$; but, as a matter of fact, a bad interpretation in the sense of necessity has driven the literature to neglect this case.

- In the case of Eq. (28), the system should evolve towards the value of Eq. (29), but in Bludman & Ross 2002, it is shown that this value is maintained only for a short period, while the asymptotic value is $w = -1$. It is in fact not difficult to show that, if the scalar field dominates, and if there is a constant asymptotic value for w, this must be -1.

- If we insert our exact solution into the expression of Γ in Steinhardt et al. 1999, we do not find consistency! The correct result is

$$\Gamma = 1 - \frac{2}{1+w}\frac{\tilde{\tilde{x}}}{(6+\tilde{x})^2} - \frac{1-w}{2\,(1+w)}\frac{\tilde{x}}{6+\tilde{x}} + 3\frac{w_B - w}{1+w}\frac{1-\Omega_\varphi}{6+\tilde{x}}\,, \tag{31}$$

where

$$x \equiv \frac{1+w}{1-w} \quad , \tilde{x} \equiv \frac{d(\log x)}{d(\log a)} \quad , \tilde{\tilde{x}} \ \equiv \frac{d\,\tilde{x}}{d(\log a)}\,, \tag{32}$$

so that, in the case when w is nearly constant, $\tilde{x}$ and $\tilde{\tilde{x}}$ can be neglected.

- It is straightforward to check that this is consistent with our solution, but what is really important is that, if w is nearly constant, Eq. (27) is not true. In the case of w nearly constant, in fact, we have from the above equation that Eq. (27) must be substituted by

$$\Gamma \approx 1 + \frac{w_B - w}{2(1+w)}(1 - \Omega_\varphi)\,, \tag{33}$$

and we find the answers to the above points.

- If w scales as w_B, then Ω_φ goes to a constant value < 1, and everything is consistent; but, if the scalar field dominates and go faster forever, then $\Omega_\varphi \to 1$, so that w can take any value. It can be indeed computed as (Copeland et al. 1998)

$$w = \frac{\lambda^2 - 3}{3}\,. \tag{34}$$

- In the case of Eq. (28), since $\Omega_\varphi \to 1$, the only way to keep Γ constant is to let $w \to -1$. In this case, the last terms in Eq. (31) cannot be neglected.

- The picture is consistent.

- Eq. (33) allows to obtain another interesting result. If there is a period in which w is nearly constant and the scalar field ultimately dominates, then the asymptotic value for Γ is 1 and the potential is *obliged* to be exponential. However, we must note that this only means that the true potential approximates an exponential in the late evolution, without saying anything else on its analytical expression.

X. CONCLUSIONS

- We presented a class of physically meaningful exponential potentials, which allow general exact solution of the Friedman and Klein-Gordon equations.

- This solution meets all the requests which are generally made for a quintessence model, except one: the possibility of switching off the acceleration in the future, which is needed for the asymptotic freedom of the model (Hellerman et al. 2001).

- This can indeed be obtained by a modification of the potential. One possibility is to bring the value of σ to the range $\sqrt{2} <$

$\sigma < \sqrt{3}$, so that the limit value for w is raised to a value greater than $-1/3$. Another possibility is to pass to a combination of exponentials (see Rubano et al. 2003).

- In exponential model, it is possible to guess a slightly positive spatial curvature which is not excluded by the present data.

- However, this model loses its validity when radiation is not negligible and a lot of interesting features occur just during that epoch. A study along these lines is presented in Franca & Rosenfeld 2002.

- This model can emulate a cosmological constant, but gives no answers about the real possibility to discriminate, at least between these two models.

- The main result which we obtain in this paper is probably pedagogical: we have learned that some statements can be misleading. Let us summarize some of them here again.

1. the most interesting epoch, in our model, is when w is mostly variable, but only with respect to $\log a$. This shows that to say

" w is necessarily almost constant" is badly stated, and risks to be merely a prejudice.

2. The study of the tracker behavior is subtle and strongly depends on the representation chosen.

3. The requirement $\Gamma > 1$ seems to be not necessary and, in any case, many widely accepted statements on exponential potentials and tracking solutions are indeed based on a result which has revealed to be incorrect.

* Let us rewrite Friedmann and Klein-Gordon equations

$$H^2 = \frac{1}{3}\left(\rho_B + \rho_\varphi\right) \quad , \dot{H} \equiv \frac{\ddot{a}}{a} - H^2 = -\frac{1}{2}\left[\rho_B\left(1+w_B\right) + \rho_\varphi\left(1+w\right)\right] , \tag{35}$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0 \,. \tag{36}$$

We then define as above

$$x \equiv \frac{\dot{\varphi}^2}{2V} = \frac{1+w}{1-w} \quad , \tilde{x} \equiv \frac{d\ln x}{d\ln a} = \frac{\dot{x}}{Hx} \quad , \quad \tilde{\tilde{x}} \equiv \frac{d^2 \ln x}{d\ln a^2} = \frac{1}{H}\frac{d}{dt}\left(\frac{\dot{x}}{Hx}\right) . \tag{37}$$

After substitution of

$$V' \equiv \frac{dV}{d\varphi} = \frac{\dot{V}}{\dot{\varphi}} \quad , V'' = \frac{dV'}{d\varphi} = \frac{1}{\dot{\varphi}}\frac{d}{dt}\left(\frac{\dot{V}}{\dot{\varphi}}\right) = \frac{\ddot{V}}{\dot{\varphi}^2} - \frac{\dot{V}\ddot{\varphi}}{\dot{\varphi}^3} \tag{38}$$

into the definition of Γ, we get

$$\Gamma = \left(\frac{\ddot{V}}{\dot{\varphi}^2} - \frac{\dot{V}\ddot{\varphi}}{\dot{\varphi}^3}\right)\frac{V\dot{\varphi}^2}{\dot{V}^2} = \frac{V\ddot{V}}{\dot{V}^2} + \frac{V}{\dot{V}\dot{\varphi}}\left(3H\dot{\varphi} + V'\right) = \frac{V\ddot{V}}{\dot{V}^2} + 3H\frac{V}{\dot{V}} + \frac{V}{\dot{\varphi}^2} \,. \tag{39}$$

Let us now compute the three terms of Γ. Starting from the last one, we have

$$\frac{V}{\dot{\varphi}^2} = \frac{1}{2x} = \frac{1-w}{2\left(1+w\right)} \,. \tag{40}$$

As for the second term, let us observe that

$$\begin{aligned}\frac{\dot{V}}{V} &= \frac{d\ln V}{dt} = \frac{d}{dt}\left(2\ln\dot{\varphi} - \log 2 - \log x\right) \\ &= \frac{2\ddot{\varphi}}{\dot{\varphi}} - \frac{\dot{x}}{x} = \frac{2}{\dot{\varphi}}\left(-3H\dot{\varphi} - V'\right) - H\tilde{x} \\ &= -6H - \frac{\dot{V}}{xV} - H\tilde{x}\,, \end{aligned} \tag{41}$$

so that we have

$$3H\frac{V}{\dot{V}} = -3\frac{1+x}{(6+\tilde{x})\,x} = -\frac{6}{1+w}\frac{1}{6+\tilde{x}}\,. \tag{42}$$

Finally, let us compute the first term

$$\frac{V\ddot{V}}{\dot{V}^2} = \frac{V}{\dot{V}}\frac{d\ln\dot{V}}{dt}\,. \tag{43}$$

First, we have

$$\frac{d\ln\dot{V}}{dt} = \frac{\dot{V}}{V} + \frac{\dot{H}}{H} + \frac{\dot{x}}{x} + \frac{d\tilde{x}/dt}{6+\tilde{x}} - \frac{\dot{x}}{1+x}\,, \tag{44}$$

and, being

$$\frac{d\tilde{x}}{dt} = \frac{d}{dt}\left(\frac{\dot{x}}{Hx}\right) = H\tilde{\tilde{x}}\,, \tag{45}$$

we get

$$\frac{d\ln\dot{V}}{dt} = -\frac{6+\tilde{x}}{1+x}Hx + \frac{\dot{H}}{H} + H\tilde{x} + \frac{H\tilde{\tilde{x}}}{6+\tilde{x}} - \frac{\tilde{x}}{1+x}Hx\,. \tag{46}$$

From this relation, together with Eq.(42), we obtain

$$
\begin{aligned}
\frac{V\ddot{V}}{\dot{V}^2} &= 1 - \frac{\dot{H}}{H^2}\frac{1+x}{(6+\tilde{x})\,x} - \frac{1+x}{x}\frac{\tilde{x}}{6+\tilde{x}} - \frac{1+x}{x}\frac{\tilde{\tilde{x}}}{(6+\tilde{x})^2} + \frac{\tilde{x}}{6+\tilde{x}} \\
&= 1 - \frac{\dot{H}}{H^2}\frac{2}{1+w}\frac{1}{6+\tilde{x}} - \frac{1-w}{1+w}\frac{\tilde{x}}{6+\tilde{x}} - \frac{2}{1+w}\frac{\tilde{\tilde{x}}}{(6+\tilde{x})^2}\,, \qquad (47)
\end{aligned}
$$

and, considering that

$$
\frac{2x}{1+x} = 1 + w\,, \qquad (48)
$$

we find

$$
\Gamma = 1 - \frac{2}{1+w}\frac{\tilde{\tilde{x}}}{(6+\tilde{x})^2} - \frac{1-w}{1+w}\frac{\tilde{x}}{6+\tilde{x}} - \frac{6}{1+w}\frac{1}{6+\tilde{x}} - \frac{\dot{H}}{H^2}\frac{2}{1+w}\frac{1}{6+\tilde{x}} + \frac{1}{2\,(\ldots} \qquad (49)
$$

We have now to eliminate the term $\dot{H}/H^2$. From Eq. (35) it is

$$
\begin{aligned}
\frac{\dot{H}}{H^2} &= -\frac{3}{2}\frac{\rho_B(1+w_B)+\rho_\varphi(1+w)}{\rho_B+\rho_\varphi} = -\frac{3}{2}\left(1+\frac{\rho_B w_B+\rho_\varphi w}{\rho_B+\rho_\varphi}\right) \\
&= -\frac{3}{2}\left[1+(1-\Omega_\varphi)\,w_B+\Omega_\varphi w\right] = \frac{3}{2}\left[\Omega_\varphi\,(w_B-w)-(1+w_B)\right]\,, \qquad (50)
\end{aligned}
$$

and, by substitution, we eventually get

$$
\Gamma = 1 - \frac{2}{1+w}\frac{\tilde{\tilde{x}}}{(6+\tilde{x})^2} - \frac{1-w}{2\,(1+w)}\frac{\tilde{x}}{6+\tilde{x}} + 3\frac{w_B-w}{1+w}\frac{1-\Omega_\varphi}{6+\tilde{x}}\,, \qquad (51)
$$

which should be compared with the formula reported in Steinhardt et al. 1999.

$$\Gamma = 1 + \frac{w_B - w}{2(1+w)} - \frac{1 + w_B - 2w}{2\,(1+w)}\frac{\tilde{x}}{6+\tilde{x}} - \frac{2}{1+w}\frac{\tilde{\tilde{x}}}{(6+\tilde{x})^2}\,. \quad (52)$$

	Λ-term $-\log(LH) = 765.3$			Exp. pot. $-\log(LH) = 767.3$		
par.	best fit	lower	upper	best fit	lower	upper
$\Omega_b h^2$	0.0226	0.0206	0.0256	0.023	0.0213	0.0266
$\Omega_{dm} h^2$	0.120	0.103	0.139	0.110	0.094	0.134
n_s	0.960	0.914	1.05	0.948	0.905	1.04
Ω_m	0.298	0.222	0.379	0.298	0.232	0.383
z_{re}	12.1	2.57	24.0	12.6	2.50	23.6
h	0.692	0.643	0.770	0.669	0.628	0.729