Higher Order Gravity Tensors

In four dimensions, field equations for gravity are taken to be

$$G_{ab} + \Lambda g_{ab} = 0$$

since this is the most general tensor which

1. is symmetric
2. depends only on the metric and its first two derivatives
3. is divergence free

(4. is linear in second derivatives of the metric)

But in five dimensions, the second order Lovelock tensor

$$H_{ab} = RR_{ab} - 2R_{ac}R^{c}{}_{b} - 2R^{cd}R_{acbd} + R_{a}{}^{cde}R_{bcde} - \frac{1}{4}g_{ab}\mathcal{L}_{\mathrm{GB}}$$

which can be obtained from variation of action containing Gauss-Bonnet term

$$\mathcal{L}_{\mathrm{GB}} = R^2 - 4R_{ab}R^{ab} + R^{abcd}R_{abcd}$$

also satisfies required conditions.

Thus should take 5-d field equations to be

$$G_{ab} + 2\alpha H_{ab} + \Lambda g_{ab} = 0$$

Should also consider extra term since $\mathcal{L}_{\mathrm{GB}}$ appears as quantum gravity correction to string theory tree level effective action.

Particularly relevant for brane world models, since they are motivated by string theory.

If dilaton is included, general second order term is instead

$$\mathcal{L}_2 = c_1\mathcal{L}_{\mathsf{GB}} - 16c_2 G_{ab}\nabla^a\phi\nabla^b\phi + 16c_3(\nabla\phi)^2\nabla^2\phi - 16c_4(\nabla\phi)^4$$

Not all coefficients fixed by string theory.

In fact low energy string theory action also suggests other possibilities:

$$R^2 \ , \quad R_{ab}R^{ab} \ , \quad R(\nabla\phi)^2 \ , \quad R\nabla^2\phi$$

However these all give ghosts at high energy (but not actually ruled out for string theory since dealing with low energy effective action).

General bulk action

$$S_{\mathsf{Bulk}} = \frac{M^3}{2}\int d^5x\sqrt{-g}e^{-2\phi}\left\{R - 4\omega(\nabla\phi)^2 + M^{-2}\mathcal{L}_2 - 2\Lambda\right\}$$

Coefficients can be determined from origin of ϕ.
For toroidal compactification of

$$\int d^{5+N}x\sqrt{-g_{(5+N)}}\left(R^{(5+N)} + c_1 M^{-2}\mathcal{L}_{\mathsf{GB}}^{(5+N)} - 2\Lambda\right)$$

with $ds^2_{5+N} = g_{ab}(x)dx^a dx^b + e^{-4\phi(x)/N}\eta_{AB}dX^A dX^B$

Obtain $\omega = -1 + N^{-1}$ and

$$c_2 = -\omega c_1 \ , \quad c_3 = (2\omega+1)\omega c_1 \ , \quad c_4 = -(2\omega+1)\omega^2 c_1$$

$\omega = -1$ corresponds to dilaton (with extra symmetries).

Define $\alpha = c_1/M^2$ for later convenience.
In string theory context $\alpha \propto \alpha'$, hence small.

As with Einstein-Hilbert action, need boundary term to get well defined action [Myers]

$$\mathcal{L}_{\rm GB}^{(b)} = 4KK_{ac}K^{ac} - \frac{8}{3}K_{ac}K^{cb}K^a{}_b - \frac{4}{3}K^3 - 8G_{ab}^{(4)}K^{ab}$$

With higher order scalar field terms as well, brane contribution to action is

$$S_{\rm brane} = -M^3 \int d^4x \sqrt{-h} e^{-2\phi} \left\{ 2K + M^{-2}\mathcal{L}_2^{(b)} + T \right\}$$

with

$$\begin{aligned} \mathcal{L}_2^{(b)} &= c_1 \mathcal{L}_{\rm GB}^{(b)} - 16c_2(K_{ab} - Kh_{ab})D^a\phi D^b\phi \\ &\quad - 16c_3(n\cdot\nabla\phi)\left(\frac{1}{3}(n\cdot\nabla\phi)^2 + (D\phi)^2\right) \end{aligned}$$

Variation of action gives generalised Israel junction conditions (as before). These do not depend on brane thickness (not true for other second order terms).

Can also derive by treating brane as δ-function contribution to energy-momentum tensor.

E.g. for $ds_5^2 = e^{2A}ds_4^2 + dz^2$ have

$$A'' \propto 2\delta(z)\ , \quad A' \propto {\rm sign}(z)$$

Field equations give

$$-3A'' + 12\alpha A'^2 A'' + \cdots = T\delta(z)$$

So for $\alpha = 0$, $T = -6A'$.

If $\alpha \neq 0$, not clear, since value of A'^2A'' is ambiguous.

Literature contains conflicting results.

Linearised Brane World Gravity

For effective 4-d gravity need to worry about perturbations of brane position as well as bulk metric.

Can address this by also perturbing coordinates, or by using gauge in which brane stays where it is.

Consider general perturbation of Randall-Sundrum-like brane world with scalar field

$$ds^2 = e^{-2k|z|}(\eta_{\mu\nu} + \gamma_{\mu\nu})dx^\mu dx^\nu + 2v_\mu dx^\mu dz + (1+\psi)dz^2$$

$$\phi = \phi_0 - \sigma|z| + \varphi$$

where $\gamma_{\mu\nu}$, v_μ, ψ and φ are small.

Require $(G_{ab} + \cdots)n^b = 0$.
Brane then remains at $z = 0$.
Equivalent of 'brane-bending' effects are included in lapse function, ψ, and shift vector, v_μ.

Useful to split $\gamma_{\mu\nu}$ into tensor and scalar parts

$$\gamma_{\mu\nu} = \bar{\gamma}_{\mu\nu} + \frac{1}{4}\gamma\eta_{\mu\nu} + \frac{4}{3}c_\chi\left(\frac{1}{4}\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box_4}\right)\chi$$

where $\gamma = \eta^{\mu\nu}\gamma_{\mu\nu}$, $c_\chi \equiv c_\chi(k,\sigma)$ and $\chi \propto 8k\varphi - \sigma\gamma$.

No brane bending gives (for $z > 0$)

$$\psi = \frac{2}{\sigma}\partial_z(\chi - \varphi) \quad , \quad v_\mu = \frac{1}{\sigma}\partial_\mu(\chi - \varphi) \quad , \quad \partial^\mu\bar{\gamma}_{\mu\nu} = 0 \ .$$

Bulk graviton equation

$$\mu_\gamma \left(\partial_z^2 - 2(2k-\sigma)\partial_z + f_\gamma^2 e^{2kz}\Box_4\right)\bar{\gamma}_{\mu\nu} = 0$$

Where $\mu_\gamma = 1 - 4[c_1 k(k-4\sigma) + 2c_2\sigma^2]$

If $\mu_\gamma(\sigma,k) < 0$ or $f_\gamma^2(\sigma,k) < 0$, bulk will have ghosts. (Does not occur if higher order gravity terms absent.)

The brane junction conditions imply

$$\mu_\gamma \partial_z \bar{\gamma}_{\mu\nu} + 4c_1[k-2\sigma]\Box_4 \bar{\gamma}_{\mu\nu} \propto -\left\{S_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box_4}\right)S\right\}$$

($S_{\mu\nu}$ is brane energy momentum tensor)

If $4c_1[k-2\sigma] < 0$, then either $M_{\mathrm{Pl}}^2 < 0$ or vacuum has non-trivial solutions with spacelike momenta (tachyons).

Similar types of instabilities occur for scalar perturbations.

E. g. for $\omega = -1$, $c_i = \alpha$ (dilaton)

If no higher order curvature terms ($\alpha = 0$), solutions have $k = 0$.

Perturbations do not have correct small momentum (large distance) behaviour (no zero mode). Hence do not get correct 4-d gravity.

Solution still valid if $\alpha > 0$.

But now have $4\alpha[k-2\sigma] < 0$ so system is unstable.

When $\alpha \neq 0$ two new solution branches appear

$$k = \sigma \pm \sqrt{\frac{1}{12\alpha} + \frac{\sigma^2}{3}}$$

(for $\omega = -1$, $c_i = \alpha$)

Bulk graviton equation

$$8\alpha k(k-\sigma)\left(\partial_z^2 - 2(2k-\sigma)\partial_z + f_\gamma^2 e^{2kz}\Box_4\right)\bar{\gamma}_{\mu\nu} = 0$$

where $f_\gamma^2 = (k-2\sigma)/(k-\sigma)$.

Graviton junction condition (take $M = 1$, $\phi_0 = 0$)

$$4\alpha(k-\sigma)\left\{2k\partial_z\bar{\gamma}_{\mu\nu} + f_\gamma^2\Box_4\bar{\gamma}_{\mu\nu}\right\} = -\left\{S_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box_4}\right)S\right\}$$

Again, ghosts and tachyons are possible.

(−)-branch is always unstable.

(+)-branch stable if $\sigma < 1/\sqrt{8\alpha}$, (so $k > 2\sigma$).

Bulk solution (Fourier space, spacelike p):

$$\bar{\gamma}_{\mu\nu} \propto e^{-ip\cdot x} e^{(2k-\sigma)z} K_{2-\sigma/k}\left(f_\gamma p e^{kz}/k\right)$$

For $p \ll k/f_\gamma$ (larger distances)

$$\partial_z\bar{\gamma}_{\mu\nu} \approx \frac{f_\gamma^2}{2(k-\sigma)}\Box_4\bar{\gamma}_{\mu\nu}$$

So 4-d gravity is obtained at larger distances.
(similar to Randall Sundrum scenario).

Extra $\Box_4\bar{\gamma}_{\mu\nu}$ term in junction conditions gives
4-d gravity at short distances too
– hence higher order gravity leads to weaker constraints.

Scalar bulk equation

$$4\alpha(2\sigma - 3k)^2 \left(\partial_z^2 - 2(2k-\sigma)\partial_z + f_\chi^2 e^{2kz}\Box_4\right)\chi = 0$$

$f_\chi^2 = 3k/(3k-2\sigma)$

Junction conditions

$$(3k-2\sigma)[12k^2(k-\sigma) + \alpha(3k-2\sigma)]\partial_z\chi + 12k^2\sigma\Box_4\chi = \frac{S}{16}$$

$$-4\alpha\sigma k(3k-2\sigma)\partial_z\chi + 12\alpha k(k-\sigma)\Box_4\chi = \Box_4\varphi$$

Qualitatively similar to graviton equations (stable for (+)-branch), but all coefficients different.

Degeneracy between scalar and tensor modes is broken. In particular can have $f_\gamma \ll f_\chi$ if σ near $1/\sqrt{8\alpha}$.

Obtain (to leading order) Brans-Dicke gravity on brane.

$$\mathcal{G}_{\mu\nu} - 2(\eta_{\mu\nu}\Box_4 - \partial_\mu\partial_\nu)\tilde{\varphi} \approx M_{\rm Pl}^{-2}S_{\mu\nu} \ , \quad -2\Box_4\tilde{\varphi} \approx M_\phi^{-2}S$$

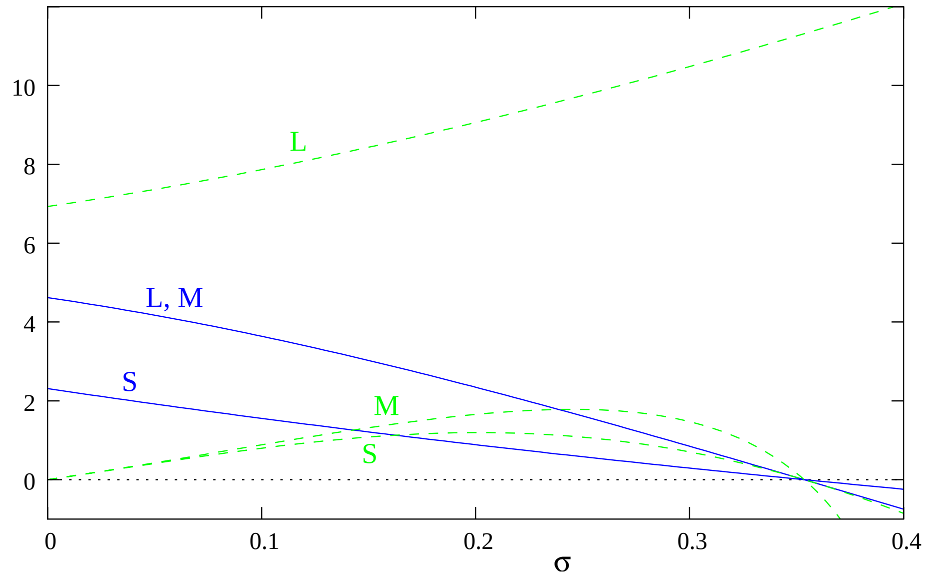
($\mathcal{G}_{\mu\nu}$ is linearised 4-d Einstein tensor, $\tilde{\varphi}$ is effective 4-d scalar) $M_{\rm Pl}$ and M_ϕ will be distance dependant.

If $p \ll k/f_\chi$ (large distances), find

$$M_{\rm Pl}^2 = 8\alpha f_\gamma^2(2k-\sigma) \ , \quad M_\phi^2 = 8\alpha(3k-2\sigma)$$

so $M_\phi \gg M_{\rm Pl}$ if solution fine-tuned to have $f_\gamma \ll 1$. Hence can potentially avoid conflict with constraints from solar system.

At medium ($k/f_\gamma \gg p \gg k/f_\chi$) and short ($p \gg k/f_\gamma$) distance scales, find $M_\phi^2 \leq 3M_{\rm Pl}^2$. But not a problem, since short distance constraints are weak.



M_{Pl}^2 and M_ϕ^2 (for $\alpha = M = e^{\phi_0} = 1$)
at (L)arge, (M)edium and (S)hort distances.

Summary

- In 5-d gravity (e.g. brane world) natural to include second order Gauss-Bonnet curvature term.
- Field equations have more solutions.
- Several new types of instabilities can occur.
- Higher order terms can effect graviton and scalar perturbations differently – can produce hierarchy between scalar and graviton couplings.
- Far weaker constraints from gravity experiments.
- Extra terms do not fix problems with Einstein gravity solution (actually destabilise it).