

The Fractal Structure of Inflationary Spacetime

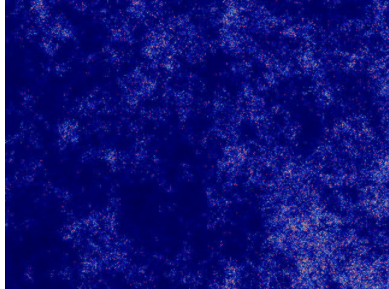
Serge Winitzki, LMU München

Rencontres de Moriond 2004

Abstract

Models of eternal inflation predict a stochastic self-similar geometry of spacetime and allow the existence of regions that never thermalize. I explore the fractal geometry of the resulting universe, concentrating on a simple model of bubble nucleation in de Sitter spacetime. I obtain the corresponding results in models of inflation driven by a scalar field. The fractal set of eternally inflating points is the focus of the investigation. The fractal dimension of the eternally inflating set, as well as the probability for eternal inflation to be realized, are found from elementary considerations in the simple model and from suitable diffusion equations in the case of scalar field inflation. The fractal dimension gives a bound on the connectivity of thermalized domains. I show that thermalized domains do not merge in generic models of scalar field-driven inflation but may merge in some brane-world inflationary scenarios.

The Fractal Structure of Inflationary Spacetime



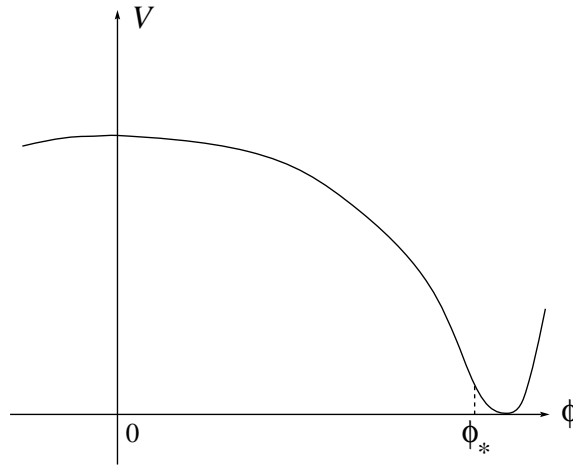
- Cosmological inflation
- Eternal inflation: self-reproducing space-time
- A toy model of eternal inflation (de Sitter bubbles)
- The eternal fractal set and its fractal dimension
- The probability to realize eternal inflation
- Stochastic description of spacetime
- Results for scalar field-driven inflation
- Connectivity of thermalized domains
- Summary

Cosmological Inflation

Models with one scalar field ϕ in Friedmann-Robertson-Walker spacetime:

$$\mathcal{L} = \frac{1}{2} \nabla_{\mu} \phi \nabla^{\mu} \phi - V(\phi); \quad ds^2 = dt^2 - a^2(t) d\mathbf{x}^2$$

Potential for the scalar field $\phi(t)$:



Evolution in the “slow roll” approximation:

$$\left(\frac{1}{a} \frac{da}{dt} \right)^2 \equiv H(\phi)^2 \approx \frac{8\pi}{3} V(\phi); \quad \frac{d\phi}{dt} \approx -\frac{V'(\phi)}{3H} = -\frac{H'(\phi)}{4\pi}$$

Exponential expansion: $H(t) \approx \text{const} \Rightarrow a(t) \sim \exp(Ht)$

Inflation as a Random Walk

- Quantum fluctuations of ϕ generate "jumps" on top of the "slow roll"
- Langevin equation for volume-averaged field ϕ :

A. Starobinsky et al. 1986:

$$\frac{d\phi}{dt} = -\frac{H'(\phi)}{4\pi} + \xi(\mathbf{x}, t); \quad \langle \xi(\mathbf{x}, t) \xi(\mathbf{x}', t') \rangle \approx \frac{1}{4\pi^2} H^3 \delta(t - t')$$

S.W. & A. Vilenkin 1999:

$$\langle \xi(\mathbf{x}, t) \xi(\mathbf{x}', t) \rangle \propto |\mathbf{x} - \mathbf{x}'|^{-4} \text{ for } H |\mathbf{x} - \mathbf{x}'| \gg 1$$

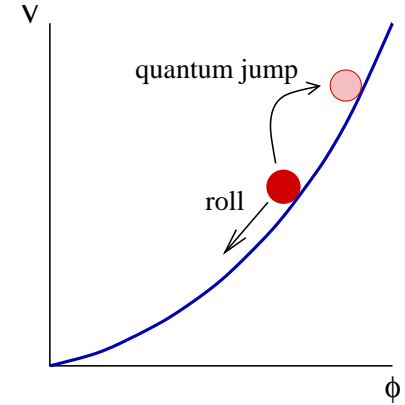
$$\langle \xi(\mathbf{x}, t) \xi(\mathbf{x}, t') \rangle \propto \exp(-2H |t - t'|) \text{ for } H |t - t'| \gg 1$$

\Rightarrow Random walk of ϕ in independent domains of size $\gtrsim H^{-1}$ on timescales $\gtrsim H^{-1}$

- Inhomogeneous spacetime metric determined by $\phi(\mathbf{x}, t)$:

$$ds^2 = dt^2 - a^2(\mathbf{x}, t) d\mathbf{x}^2$$

- Inflation ends at $\phi = \phi_*$ ("thermalization") at some time $t_*(\mathbf{x})$
- Inflation does not finish everywhere at the same time



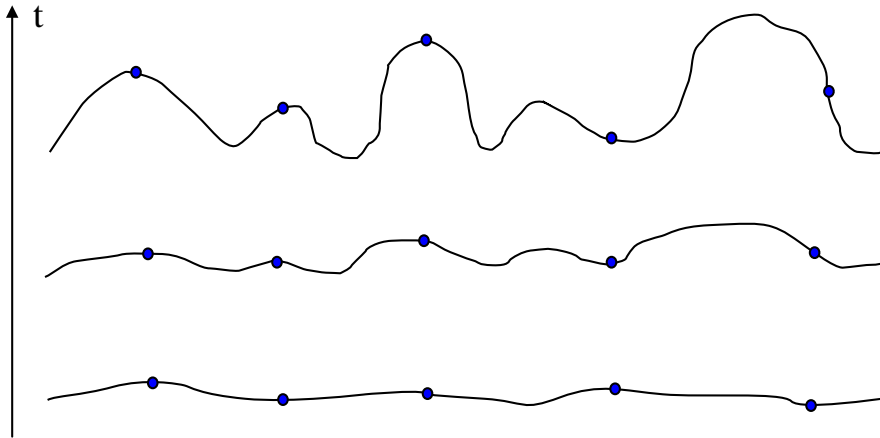
Eternal Inflation

- Fluctuating values of ϕ result in a *locally* FRW spacetime:

$$ds^2 = dt^2 - a^2(t, x) dx^2, \quad a(t, x) \approx \exp \left[\int H(t, x) dt \right]$$

- Significant variations of a and H occur on scales $L \gtrsim H^{-1}$

Sketch of equal- ϕ hypersurfaces:

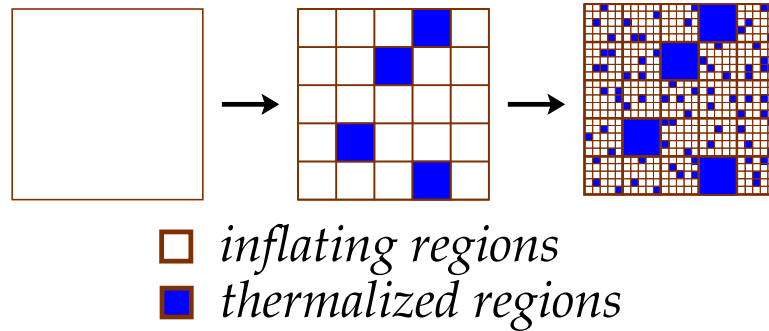


Comoving observers (•) find different expansion factors

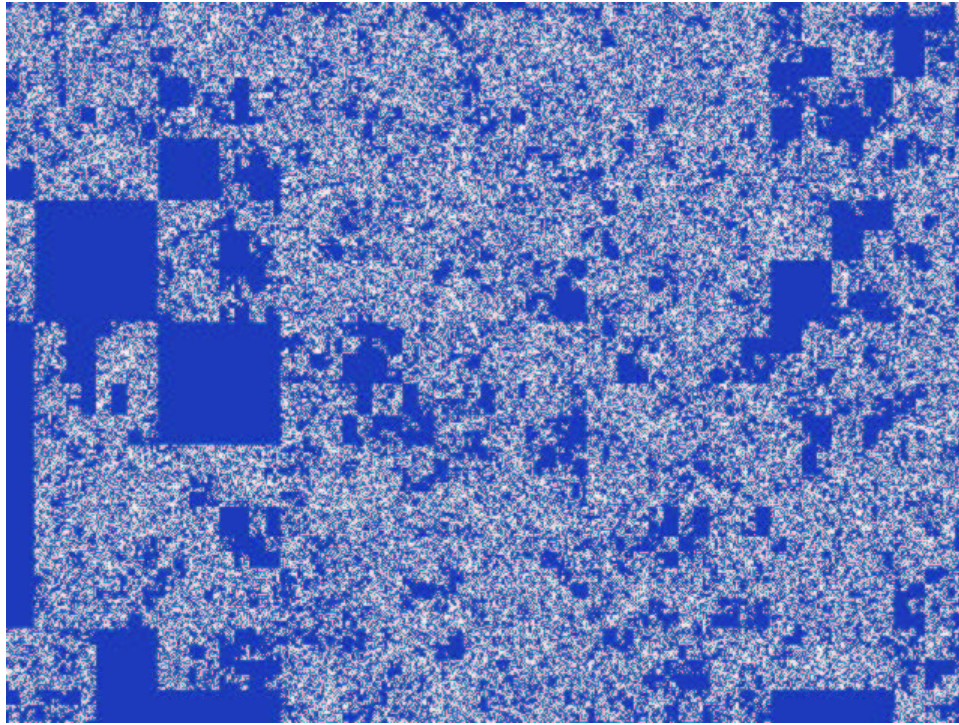
- Jumps $\delta\phi \sim \frac{1}{2\pi}H$ dominate over slow roll if $|\dot{\phi}| \ll \frac{1}{2\pi}H^2$
- Random walk of $\phi \Rightarrow$ “self-reproducing Universe”
- Due to exponentially faster expansion, inflation lasts arbitrarily long in *some* regions

Eternal Inflation in a Box

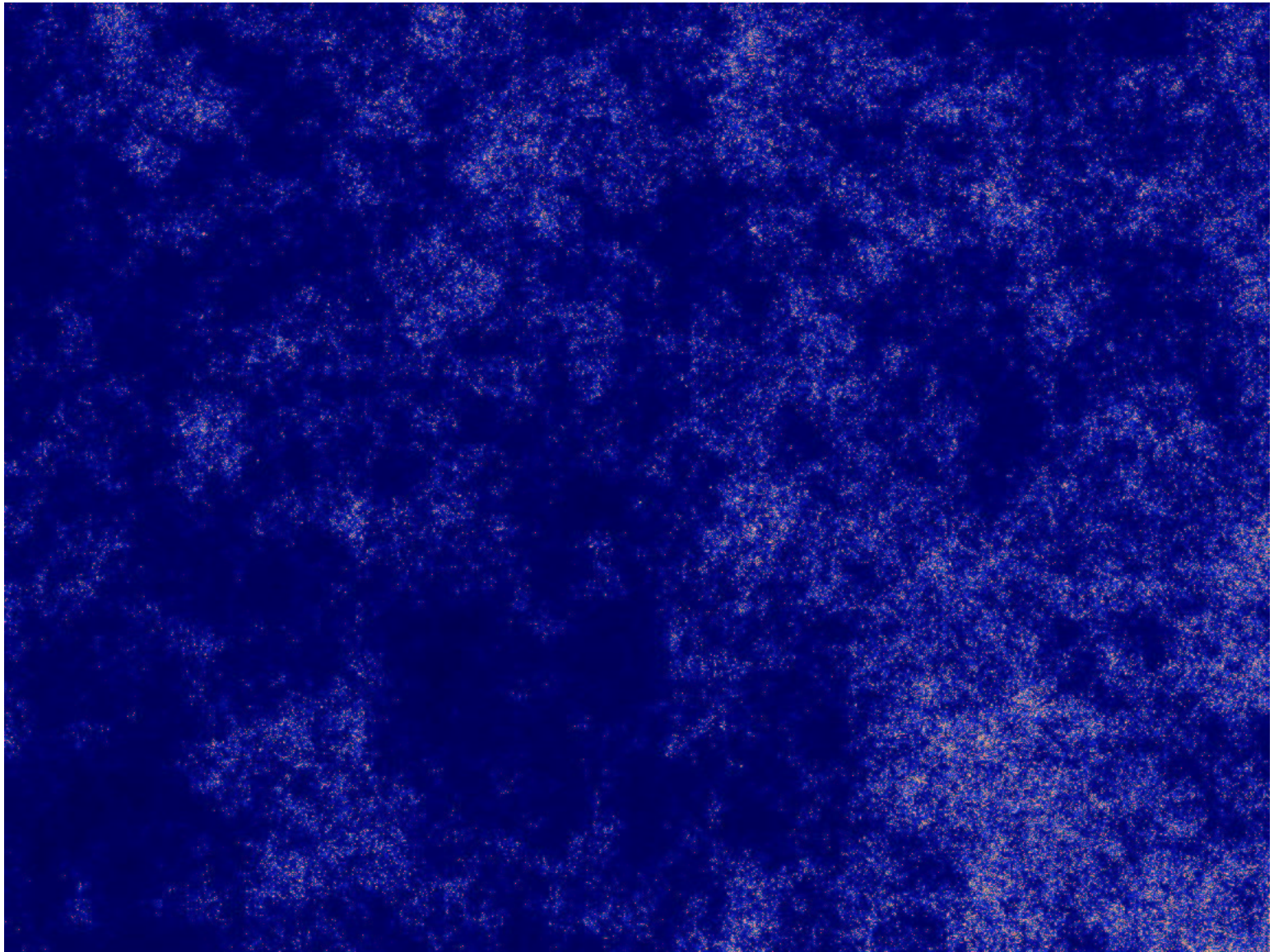
- Simulation of a discrete spacetime in 2+1 dimensions
 - Start with an inflating square (painted white) and:
 - * With probability p : divide into $N \times N$ inflating squares
 - * With probability $1 - p$: paint in blue (thermalize)
 - Repeat for each inflating square *ad infinitum*



- Imitates bubble nucleation in de Sitter space, viewed in comoving coordinates:
 - nucleation probability per horizon volume = $1 - p$
- Have eternal inflation iff $N^2 p > 1$
- The area of white squares $\rightarrow 0$
- After infinite time, consider the set E of eternally white points (“eternal set”)



Eternal set E (white points) approximated on a grid



Approximation to the eternal set E (white points) for inflation

Definition of “Eternal Fractal”

“Eternity is a very long time, especially toward the end” – Anon.

S.W. 2002

- If eternal inflation takes place, then:
 - Every comoving observer reaches the end of inflation with *probability 1*
 - At any time, the volume of the inflating domain is exponentially growing
- “Eternal points”: comoving world-lines that never finish inflation
 - Eternal points exist with nonzero probability in each horizon volume

⇒ Consider the “eternal set” E on an initial spacelike slice

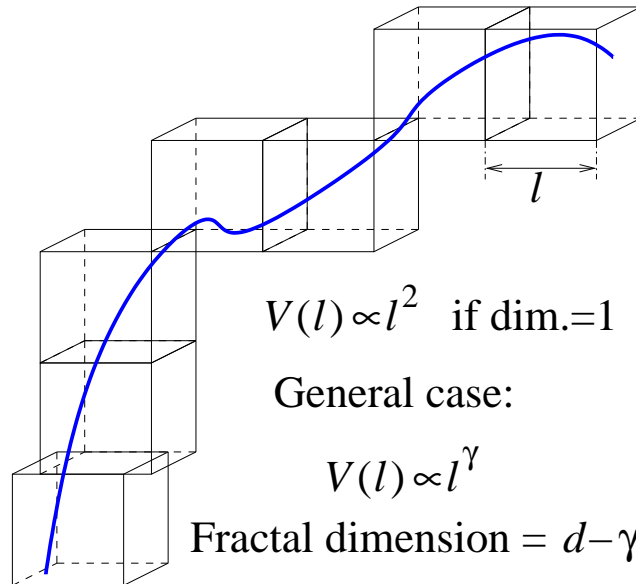
- Definition and properties of E are coordinate-independent
- The set E has fractal structure due to self-reproduction (watch the movie now)
- Questions:
 - Probability to have eternal points $\neq 0$? (= “density” of the set E)
 - Geometry of E : fractal dimension = ?
 - Does E contain thermalized domains in its interior?

Fractal Dimension

- Definition of fractal dimension:

- Take a set E embedded in a d -dimensional space
- Coarse-grain the set E on scales l by a rectangular lattice
- Find the coarse-grained volume $V_l(E)$
 - * only need the asymptotic of $V_l(E)$ at $l \rightarrow 0$, e.g. $V_l(E) \sim l^\gamma$
- Fractal dimension is defined by

$$\dim E \equiv d - \lim_{l \rightarrow 0} \frac{\ln V_l(E)}{\ln l} = d - \gamma$$



Results for the Discrete Model

- What is the probability to realize eternal inflation?

- it is the same as the probability for having a nonempty set E

- Probability to have *no* eternal points in the initial region (of size 1):

$$X \equiv \text{Prob}(\text{no e.p.}) = 1 - p + pX^{N^2}; \quad X = 1 \text{ or } X = X_0(p, N)$$

the second solution X_0 exists if $pN^2 > 1$

- Probability to *have* eternal points in a domain of size $L \ll 1$:

$$1 - X(L) = (1 - X) p^{n(L)}; \quad LN^{n(L)} = 1$$

$$\Rightarrow 1 - X(L) = (1 - X) L^\gamma, \quad \gamma \equiv -\frac{\ln p}{\ln N} > 0$$

- Coarse-grained volume of E at scale L :

$$V(L) = V_0 X L^\gamma \propto L^\gamma$$

\Rightarrow Fractal dimension of the set E : $\dim E = 2 - \gamma = 2 + \ln p / \ln N$

- If $pN^2 > 1$ then $\dim E > 0$ (condition to realize eternal inflation)
- If $\dim E < 2$, then *all* thermalized domains merge

Stochastic Description of Inflation

- Langevin equation for the random walk of ϕ :

$$\phi(t + \Delta t) - \phi(t) = v(\phi) \Delta t + \xi(x, t) \sqrt{2D(\phi) \Delta t}$$

- The comoving-space distribution $P(\phi, t)$ satisfies the Fokker-Planck equation:

$$\frac{\partial P(\phi, t)}{\partial t} = \frac{\partial^2}{\partial \phi^2} (DP) - \frac{\partial}{\partial \phi} (vP); \quad D \equiv \frac{H^3(\phi)}{8\pi^2}; \quad v \equiv -\frac{H'(\phi)}{4\pi}$$

Asymptotic solution at late times $t \gg H^{-1}$ is $P(\phi, t) \propto \exp(-\gamma t) P^{(0)}(\phi)$ with $\gamma > 0$

- Every observer finishes inflation with probability 1 since $\gamma > 0$

A. Linde, et al. 1994

- Volume expansion $\sim a^3(t) \Rightarrow$ the volume distribution $P_V(\phi, t)$ satisfies the equation

$$\frac{\partial P_V(\phi, t)}{\partial t} = \frac{\partial^2}{\partial \phi^2} (DP_V) - \frac{\partial}{\partial \phi} (vP_V) + 3HP_V$$

Asymptotic solution at late times $t \gg H^{-1}$:

$$P_V(\phi, t) \propto \exp(\tilde{\gamma} t) P_V^{(0)}(\phi)$$

- Can compute γ and $\tilde{\gamma}$ from $V(\phi)$, but γ and $\tilde{\gamma}$ depend on the choice of time coordinate t
- Eternal inflation occurs if $\tilde{\gamma} > 0$ (generically this is so)

Results for Scalar-Field Inflation

- Distribution of ϕ for comoving points, using the scale factor as time variable, $t^{(0)} \equiv \ln a$:

$$\frac{\partial P(\phi, t^{(0)})}{\partial t} = \frac{\partial^2}{\partial \phi^2} (D^{(0)} P) - \frac{\partial}{\partial \phi} (v^{(0)} P); \quad D^{(0)} \equiv \frac{H^2}{8\pi^2}; \quad v \equiv -\frac{H'}{4\pi H}$$

Asymptotic solution at late times $t^{(0)} \gg 1$:

$$P(\phi, t^{(0)}) \propto \exp(-\gamma t^{(0)}) P^{(0)}(\phi)$$

- Fractal dimension of E is: $\dim E = 3 - \gamma$
- Probability $X(\phi)$ to have *no* eternal points in a H^{-1} size region satisfies the equation:

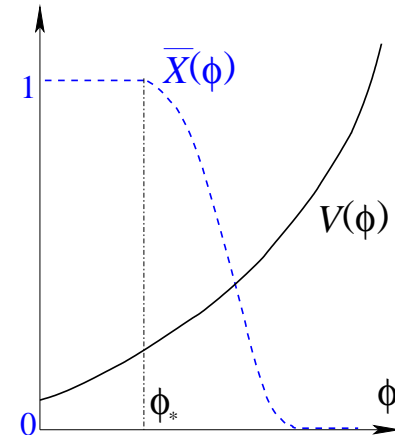
$$D \frac{\partial^2 X}{\partial \phi^2} + v \frac{\partial X}{\partial \phi} + 3HX \ln X = 0$$

Boundary conditions:

$0 < X(\phi) < 1$; $X = 1$ at thermalization;

$X = 0$ at Planck boundary

- Equation for X is coordinate-invariant
- Conditions for the existence of eternal inflation: $\tilde{\gamma} > 0 \Leftrightarrow$ a solution $X(\phi) \neq 1$ exists



--- Probability $\bar{X}(\phi) = 1 - X(\phi)$ to have eternal points

Derivation of the Equation for $X(\phi)$

- Assume probability density $P(\xi_1, \dots, \xi_N)$ for $N = e^{3H\Delta t}$ independent domains after one e-folding, starting with ϕ_0 :

$$X(\phi_0) = \int d\xi_1 \dots d\xi_N P(\xi_1, \dots, \xi_N) \prod_{i=1}^N X\left(\phi_0 + v\Delta t + \xi_i \sqrt{2D\Delta t}\right)$$

- Assume independence of domains, $P(\xi_1, \dots, \xi_N) = P(\xi_1) \dots P(\xi_N)$

$$X(\phi_0) = \left[\int d\xi P(\xi) X\left(\phi_0 + v\Delta t + \xi \sqrt{2D\Delta t}\right) \right]^{\exp(3H\Delta t)}$$

- Take $\partial/\partial\Delta t$ at $\Delta t = 0$ and Taylor expand $X(\phi)$ to $O(\Delta t^{3/2})$ using

$$\int d\xi P(\xi) = 1, \quad \int \xi d\xi P(\xi) = 0, \quad \int \xi^2 d\xi P(\xi) = 1$$

- Result:

$$\begin{aligned} \frac{\partial X(\phi_0)}{\partial \Delta t} = 0 &= 3HX \ln X + \lim_{\Delta t \rightarrow 0} \left(v + \frac{\xi \sqrt{2D}}{2\sqrt{\Delta t}} \right) \int d\xi P(\xi) X'(\phi_0 + v\Delta t + \xi \sqrt{2D\Delta t}) \\ &= DX'' + vX' + 3HX \ln X \end{aligned}$$

- Boundary condition: $X = 1$ at thermalization, $X = 0$ at Planck boundary
- Same equation gives probability for any “global absence” property of E

Are Thermalized Domains Connected?

S.W. 2004, in preparation

- If eternal set E encloses our thermalized domain in its interior, we shall never see other thermalized domains
- The set E cannot enclose any thermalized domains if its fractal dimension $\dim E < 2$
- The set E encloses thermalized domains if it contains “sheets” (*sheet percolation*)
- In the discrete model, it is proven that E percolates if $\dim E$ is sufficiently close to 3
- Sheet percolation of E is controlled by a “strong percolation” event $S(\phi)$ which satisfies

$$DS'' + vS' + 3HG(S) = 0$$

Boundary conditions: same as for $\bar{X}(\phi)$

- The existence of nontrivial solution $S(\phi) \neq 0$ guarantees the sheet percolation of E
- The sufficient condition for this is

$$3 - \dim E \equiv \gamma < \sup_{x \in [0,1]} \frac{G(x)}{x} \sim 10^{-4}$$

- Generically, the potential $V(\phi)$ should be flat $\Rightarrow \gamma \ll 1$
- $\dim E < 2$ in some brane-world scenarios, so all thermalized domains are merged

Summary

- Eternal inflation predicts a fractal structure of the spacetime on very large scales (\gg size of observable Universe)
- Defined the fractal set E of eternally inflating points
- Determined fractal dimension $\dim E$ and “density” $1 - X(\phi)$ of the eternal set
- All quantities are coordinate-invariant characteristics of the spacetime
- Sufficient condition: thermalized domains merge if $\dim E < 2$
- Thermalized regions are isolated if $\dim E$ is very close to 3 (generic in scalar-field inflation!)
- Thermalized regions are connected in some brane-world inflation scenarios ($\dim E < 2$)
- Implications for hybrid inflation and for holography