Consistent massive graviton on arbitrary background

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Fierz-Pauli theory (1939)

$$S_{h,m} = -\frac{1}{2}\bar{M}_h^2 \int d^4x \ h_{\mu\nu}\Big[\mathcal{E}^{\mu\nu\rho\sigma} + \frac{\bar{m}^2}{2}\left(g^{\rho\mu}g^{\sigma\nu} - g^{\mu\nu}g^{\rho\sigma}\right)\Big]h_{\rho\sigma}.$$

$$\boxed{\delta\bar{E}_{\mu\nu} \equiv \mathcal{E}_{\mu\nu}{}^{\rho\sigma}h_{\rho\sigma} + \frac{\bar{m}^2}{2}\Big(h_{\mu\nu} - h\,\eta_{\mu\nu}\Big) = 0}$$

- Field eqs. for a massive graviton that has 5 degrees of freedom.

  - $\partial^\nu\delta\bar{E}_{\mu\nu} \implies$ 4 vector constraints : $\partial^\mu h_{\mu\nu} - \partial_\nu h = 0.$
  - Taking another derivative : $2\partial^\mu\partial^\nu\delta\bar{E}_{\mu\nu} + \bar{m}^2\eta^{\mu\nu}\delta\bar{E}_{\mu\nu} = -\frac{3}{2}\bar{m}^4 h.$
  - Scalar constraint $\boxed{h = 0.}$

- **Only linear massive gravity theory free of ghost.**

Non-linear massive gravity

Search for a non-linear massive gravity theory with the following properties :

1. is Lorentz invariant,
2. admits flat space-time as vacuum solution,
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History

- van Dam, Veltman and Zakharov (vDVZ) discontinuity (1970) : FP do not recover GR in the massless limit,
- Vainshtein mechanism (1972) : screening mechanism to recover GR.
- Boulware Deser (BD) ghost (1972) : a 6th dof reappear in any non-linear massive gravity theory.
- de Rham, Gabadadze and Tolley (dRGT) theory (2011) : non-linear theory free of the BD ghost.

Ghost-free massive gravity theory

$$\boxed{S = M_g^2 \int \mathrm{d}^4x \sqrt{|g|} \Big[R(g) - 2m^2 V\left(S; \beta_n\right) \Big],}$$

$$V(S; \beta_n) = \sum_{n=0}^{3} \beta_n e_n(S)\,,$$

- Square-root matrix $S^{\mu}{}_{\rho} S^{\rho}{}_{\nu} = g^{\mu\rho} f_{\rho\nu}$,
- $e_n(S)$ elementary symmetric polynomials :

$$e_0(S) = 1\,, \quad e_1(S) = \mathrm{Tr}[S]\,, \quad e_2(S) = \frac{1}{2}\Big(\mathrm{Tr}[S]^2 - \mathrm{Tr}[S^2]\Big)\,,$$
$$e_3(S) = \frac{1}{6}\Big(\mathrm{Tr}[S]^3 - 3\mathrm{Tr}[S]\mathrm{Tr}[S^2] + 2\mathrm{Tr}[S^3]\Big)$$

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Absence of ghost

- in the decoupling limit (dRGT) for $f_{\mu\nu} = \eta_{\mu\nu}$,
- for the full theory (Hassan, Rosen),
- using vierbeins.

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Field equations

$$\boxed{E_{\mu\nu} \equiv \mathcal{G}_{\mu\nu} + m^2 \, V_{\mu\nu} = 0 \,,}$$

$$\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,, \qquad V_{\mu\nu} \equiv \frac{-2}{\sqrt{|g|}} \frac{\delta(\sqrt{|g|} V)}{\delta g^{\mu\nu}} \,.$$

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Linearized field equations around a background solution

$$\boxed{\delta E_{\mu\nu} \equiv \delta\mathcal{G}_{\mu\nu} + m^2 \delta V_{\mu\nu} \equiv \Big[\tilde{\mathcal{E}}_{\mu\nu}{}^{\rho\sigma} + m^2 \, \mathcal{M}_{\mu\nu}{}^{\rho\sigma} \Big] \, h_{\rho\sigma} = 0 \,,}$$

Variation of the matrix S - Sylvester equation

Cayley-Hamilton theorem

$$S^4 - e_1 S^3 + e_2 S^2 - e_3 S + e_4 \mathbb{1} = 0\,.$$
$$\Big[e_3 \mathbb{1} + e_1 S^2\Big]\,\delta S = F\big(\delta S^2\big)\,.$$

- Solution for δS iff $\mathbb{X} \equiv e_3 \mathbb{1} + e_1 S^2$ is invertible.

Sylvester equation

$$S^\mu{}_\nu\,(\delta S)^\nu{}_\sigma + (\delta S)^\mu{}_\nu\,S^\nu{}_\sigma = \delta[S^2]^\mu{}_\sigma\,.$$

- Explicit solution for δS iff S and $-S$ do not have common eigenvalues $\iff$ $\det\big(\mathbb{X}\big) \neq 0$.

The beta 1 model

Setting $\beta_2 = \beta_3 = 0$ while keeping β_0, $\beta_1 \neq 0$ and $f_{\mu\nu}$ arbitrary.

Field equations

$$\mathcal{G}_{\mu\nu} + m^2 \Big[\beta_0\, g_{\mu\nu} + \beta_1\, g_{\mu\rho}\big(e_1(S)\delta^\rho_\nu - S^\rho{}_\nu\big)\Big] = 0\,,$$

Solve for $S^\mu{}_\nu$:

$$S^\rho{}_\nu = \frac{1}{\beta_1 m^2}\left[R^\rho{}_\nu - \frac{1}{6}\delta^\rho_\nu R - \frac{m^2\beta_0}{3}\,\delta^\rho_\nu\right].$$

- Only possible in the β_1 model.
- Can be used to eliminate any occurrences of S in the linearized field equations.

Linearized field equations around arbitrary background

$$\delta E_{\mu\nu} \equiv \delta\mathcal{G}_{\mu\nu} + m^2 \delta V_{\mu\nu} \equiv \left[\tilde{\mathcal{E}}_{\mu\nu}{}^{\rho\sigma} + m^2\,\mathcal{M}_{\mu\nu}{}^{\rho\sigma}\right] h_{\rho\sigma} = 0\,,$$

- contains both the curvature of $g_{\mu\nu}$ and the matrix $S^{\mu}{}_{\nu}$,
- **In the β_1 model, we can express everything as a function of $g_{\mu\nu}$ and its curvature.**
- **We now take these equations as our starting point.**

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Counting the degrees of freedom

- 4 vector constraints : $\nabla^{\nu}\delta E_{\mu\nu} = 0$
- Scalar constraint : unlike in the F-P theory, it cannot be obtained from a linear combination of $g^{\mu\nu}\delta E_{\mu\nu}$ and $\nabla^{\mu}\nabla^{\nu}\delta E_{\mu\nu}$.

Search for a scalar constraint

All possible ways of tracing $\delta E_{\mu\nu}$ or the derivative of its divergence with $S^{\mu}{}_{\nu}$:

$$\Phi_i \equiv [S^i]^{\mu\nu}\,\delta E_{\mu\nu}\,, \qquad 0 \le i \le 3$$
$$\Psi_i \equiv [S^i]^{\mu\nu}\nabla_\nu\nabla^\lambda\,\delta E_{\lambda\mu} \qquad 0 \le i \le 3\,.$$

Find a linear combination of these 8 scalars for which the 2nd derivative terms vanish :

$$\sum_{i=0}^{3}\left(u_i\,\Phi_i + v_i\,\Psi_i\right) \sim 0,$$

More details on the search for a scalar constraint

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$$\sum_{i=0}^{3}(u_i\,\Phi_i + v_i\,\Psi_i) \sim \sum_{i=1}^{26}\alpha_i \aleph_i = 0,$$

$$\aleph_i = \{\nabla_\rho\nabla_\sigma\, h^{\rho\sigma}, ..., [S^3]^{\rho\sigma}\,[S^3]^{\mu\nu}\,\nabla_\rho\nabla_\sigma\, h_{\mu\nu}\}$$

- $\alpha_i = 0$: 26 equations for 7 unknowns $\{u_i, v_i\}$, only the trivial solution.
- All the $\aleph_i$ are not all independent from each other : non trivial identities (**syzygies**) linking them $\Longrightarrow$ Reduces the number of equations to be solved to 7.

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The fifth scalar constraint

$$\boxed{-\frac{m^2\,\beta_1\,e_4}{2}\,\Phi_0 - e_3\,\Psi_0 + e_2\,\Psi_1 - e_1\,\Psi_2 + \Psi_3 = 0\,.}$$

Applications

- In the flat space-time limit we recover $h = 0$.

Einstein space-times

- $R_{\mu\nu} = \Lambda g_{\mu\nu} \implies S^\rho{}_\nu = -\frac{\tilde{\beta}_0}{3\beta_1}\delta^\rho_\nu \,, \quad \tilde{\beta}_0 = \beta_0 - \frac{\Lambda}{m^2}$
- The scalar constraint is

$$\boxed{\frac{\tilde{\beta}_0^3}{54\beta_1^3}\left(\nabla^\mu\nabla^\nu\delta E_{\mu\nu} - \frac{m^2\tilde{\beta}_0}{6}g^{\mu\nu}\delta E_{\mu\nu}\right) = -\frac{m^4\tilde{\beta}_0^5}{648\beta_1^3}\,h\left(1 + \frac{2\Lambda}{\tilde{\beta}_0 m^2}\right) = 0}$$

Conclusion

- Theory for a massive graviton propagating in a single arbitrary background metric (β_1 model).
- Five covariant constraints for massive gravity in a metric formulation.
- Need to find the constraints in the full model (not only β_1).