Statistical Decoupling of Lagrangian Fluid Parcel in Newtonian Cosmology

Xin Wang (CITA)

ApJ, 820, 30, (2016)

The Universe is something really huge!!

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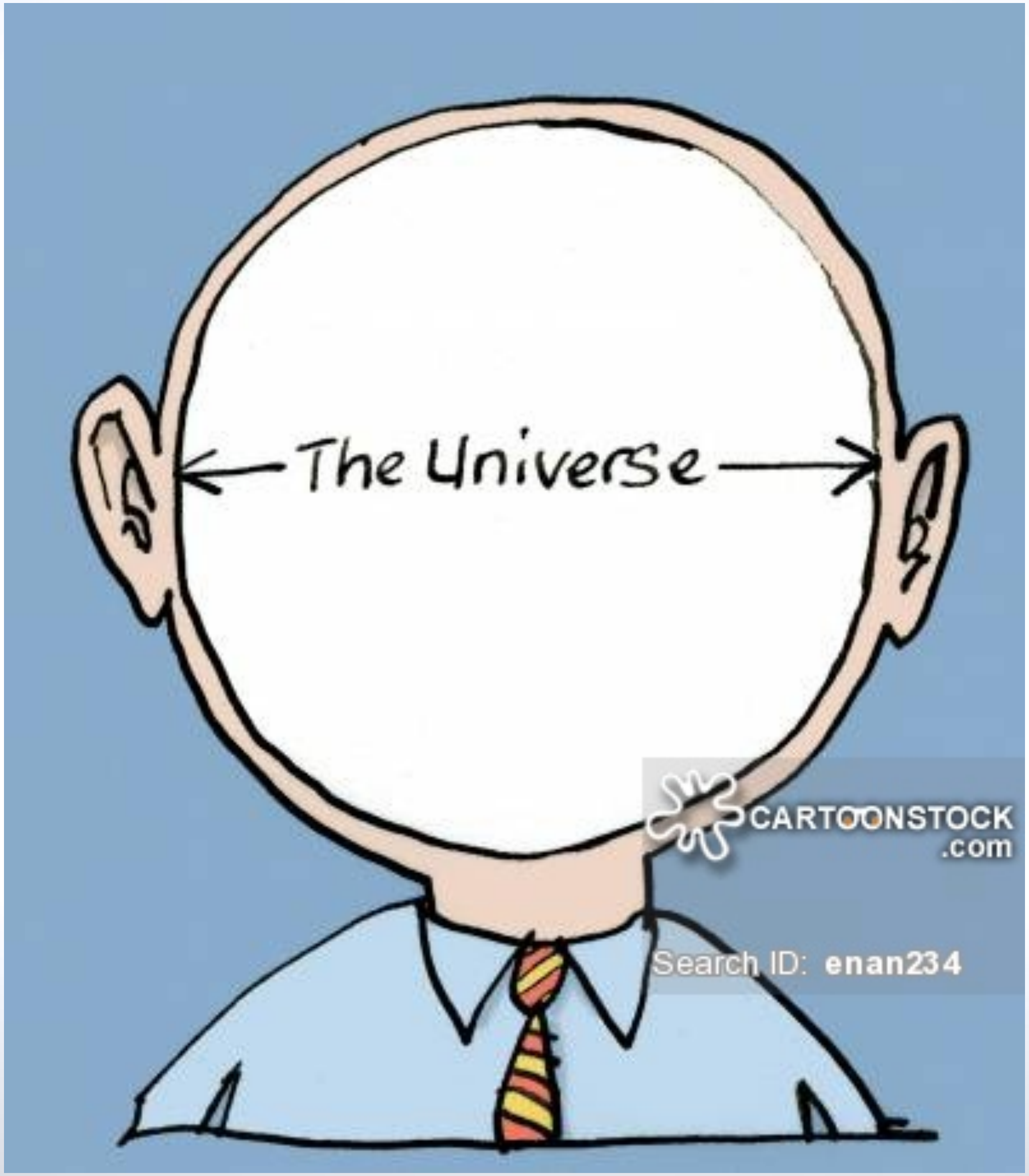
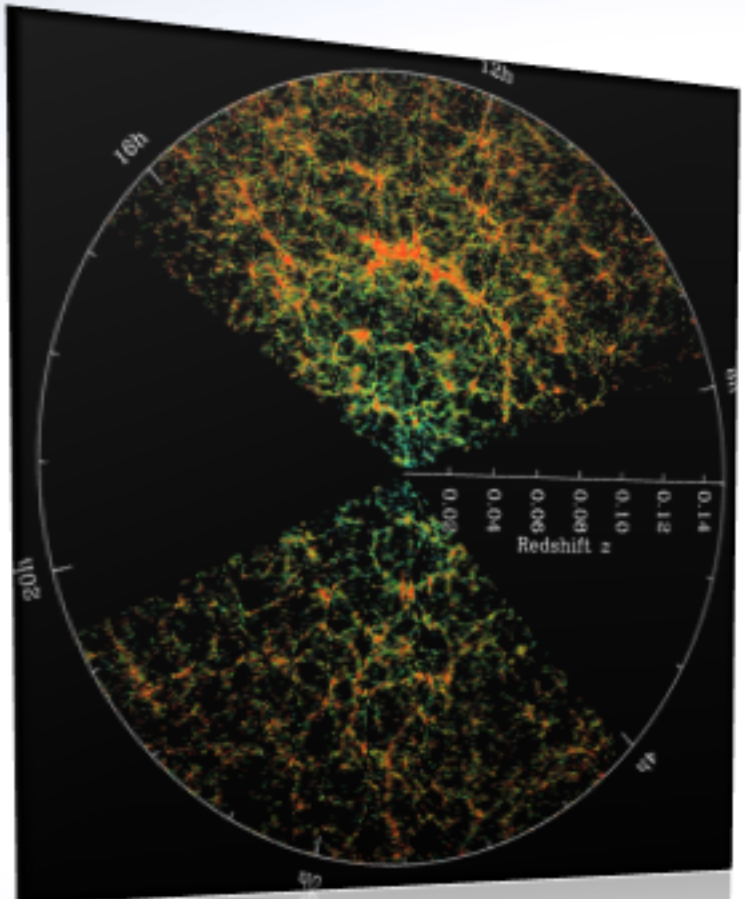
Understanding the Universe



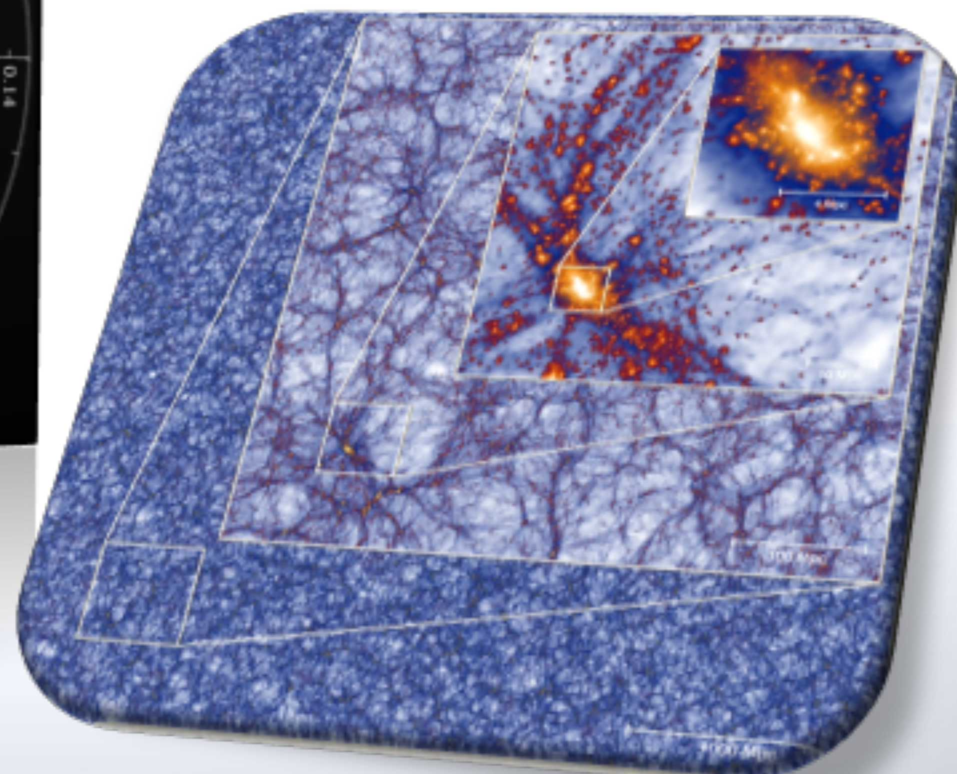
Understanding the Universe

- **So, we make simplification**

Large-scale Structure



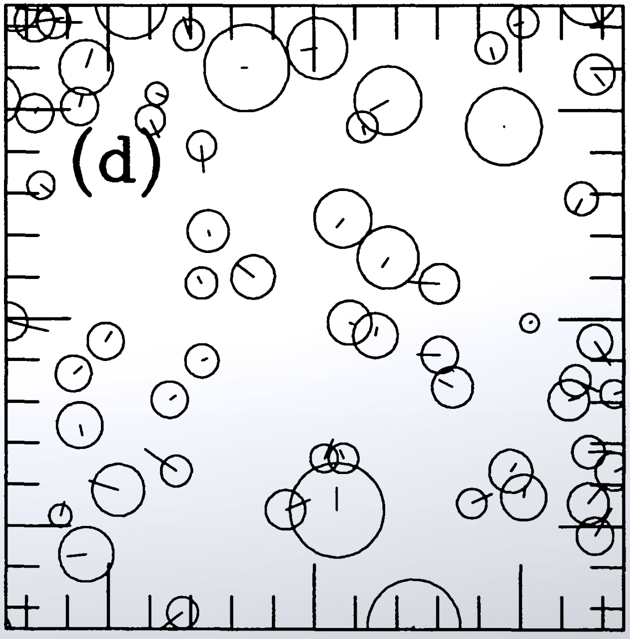
SDSS



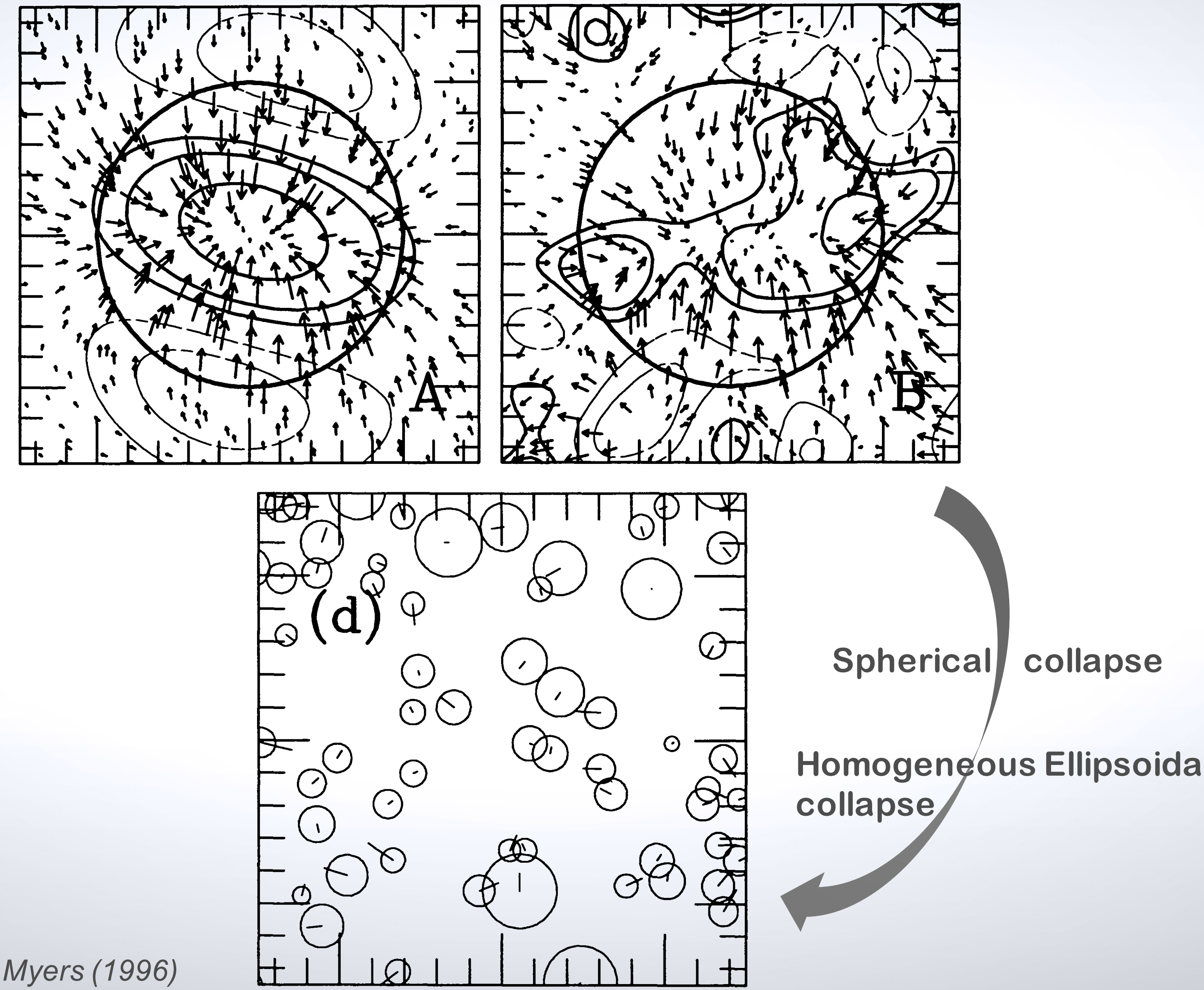
*Millennium Run (*Springel et al. 2005*)*

Individual Cosmic Object

- ***Convenient at (highly) nonlinear scale:***
  - Halos, voids, filaments, clusters, galaxies, stars …
  - their various properties: masses, ages, time evolution, etc.
    - Halo model



Individual Cosmic Object



Bond & Myers (1996)

Newtonian Gravity + Cosmic Fluid

- **Oversimplification?:**
  - *non-local, non-linear gravity*

- **Eulerian Fluid in Newtonian Gravity**

$$\frac{\partial\delta(\mathbf{x},\tau)}{\partial\tau} + \nabla\cdot\{[1+\delta(\mathbf{x},\tau)]\mathbf{v}(\mathbf{x},\tau)\} = 0\,,$$

$$\frac{\partial\mathbf{v}(\mathbf{x},\tau)}{\partial\tau} + \mathscr{H}(\tau)\mathbf{v}(\mathbf{x},\tau) + [\mathbf{v}(\mathbf{x},\tau)\cdot\nabla]\mathbf{v}(\mathbf{x},\tau) = -\nabla\Phi(\mathbf{x},\tau)\,,$$

$$\boxed{\nabla^2\Phi(\mathbf{x},\tau) = \tfrac{3}{2}\Omega\mathscr{H}^2(\tau)\delta(\mathbf{x},\tau)}\,.$$

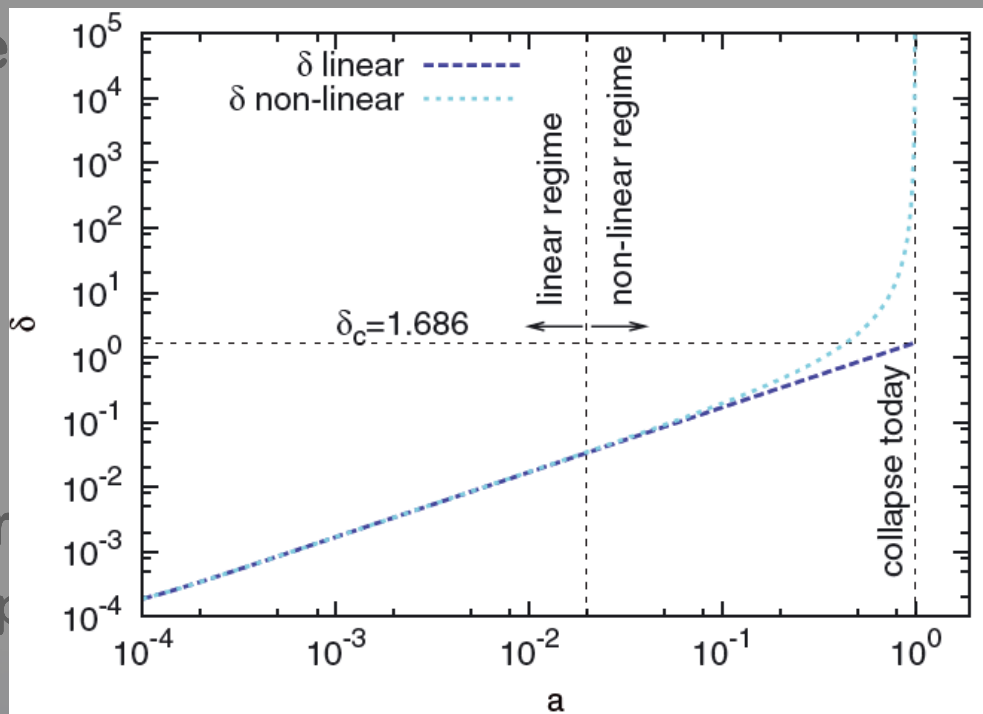
- **Lagrangian fluid dynamics**
  - Spherical collapse

$$\frac{d}{d\tau}\delta_\rho = -(1+\delta_\rho)\theta,$$

$$\frac{d}{d\tau}\theta = -\left[\mathcal{H}(\tau)\theta + \frac{1}{3}\theta^2 + \sigma^{ij}\sigma_{ij}\right] - 4\pi G_N\bar{\rho}a^2\delta_\rho,$$

Newtonian Gravity + Cosmic Fluid

- **Oversimplification?:**
  - *non-local, non-linear gravity*
- **Eule**
- **Lagr**
  - Sp



$\tau)$,

$$\frac{d}{d\tau}\delta_\rho = -(1+\delta_\rho)\theta,$$

$$\frac{d}{d\tau}\theta = -\left[\mathcal{H}(\tau)\theta + \frac{1}{3}\theta^2 + \sigma^{ij}\sigma_{ij}\right] - 4\pi G_N \bar{\rho} a^2 \delta_\rho,$$

Large-scale Structure & GR

- **Lagrangian fluid dynamics**
  - In general,

$$\frac{d}{d\tau}\delta_\rho = -(1+\delta_\rho)\theta,$$

$$\frac{d}{d\tau}\theta = -\left[\mathcal{H}(\tau)\theta + \frac{1}{3}\theta^2 + \sigma^{ij}\sigma_{ij}\right] - 4\pi G_N \bar{\rho} a^2 \delta_\rho,$$

$$\frac{d}{d\tau}\sigma_{ij} = -\left[\mathcal{H}(\tau)\sigma_{ij} + \frac{2}{3}\theta\sigma_{ij} + \sigma_{ik}\sigma^k_j - \frac{1}{3}\sigma_{mn}\sigma^{mn}\delta^K_{ij}\right] - \varepsilon_{ij},$$

  - Where velocity gradient $= \frac{\theta}{3}\delta^K_{ij} + \sigma_{ij} + \omega_{ij}$
  - and

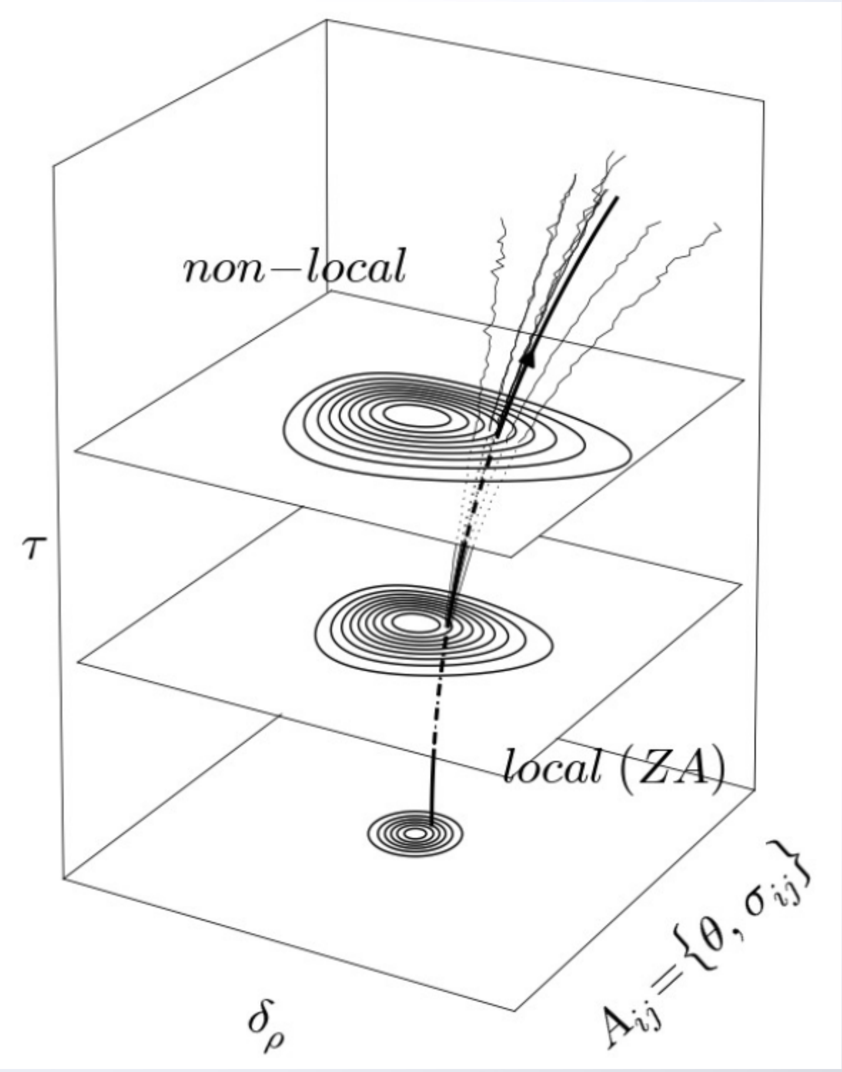
$$\Phi_{ij} = \frac{\nabla^2\Phi}{3}\delta^K_{ij} + \varepsilon_{ij} = \frac{4\pi G_N \bar{\rho} a^2 \delta_\rho}{3}\delta^K_{ij} + \varepsilon_{ij}$$

- **Tidal field**

$$\varepsilon_{ij}(\mathbf{x}) = G_N \bar{\rho} a^2 \int d^3x' \left[\frac{\delta^K_{ij}}{r^3} - 3\frac{r_i r_j}{r^5}\right]\delta_\rho(\mathbf{x}')$$

Large-scale Structure & Non-locality

- Question
  - Is SC, HEC the best we can do?
  - Is that possible to construct a nonlinear but local theory?
  - What's the best approach? (theoretically)
  - The "local approximation"
  - with data? (simulation?)
    - Measure it !!
  - some "mean effective evolution"
  - Mean?? -> *statistical description ?*



Statistical Decoupling of Lagrangian Fluid Parcel in Newtonian Cosmology

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Dynamical to PDF Evolution

- **From Dynamics to statistics**
  - **Define dynamical variables**

$$\psi = \{\delta_\rho, A_{ij}\} = \{\delta_\rho, \theta, \sigma_{ij}\}$$

  - **Lagrangian dynamical evolution**

$$\frac{d}{d\tau}\psi(\tau) = \chi[\psi, \varepsilon_{ij}]$$

  - **Which in details**

$$\frac{d}{d\tau}\delta_\rho = -(1+\delta_\rho)\theta,$$

$$\frac{d}{d\tau}\theta = -\left[\mathcal{H}(\tau)\theta + \frac{1}{3}\theta^2 + \sigma^{ij}\sigma_{ij}\right] - 4\pi G_N \bar{\rho} a^2 \delta_\rho,$$

$$\frac{d}{d\tau}\sigma_{ij} = -\left[\mathcal{H}(\tau)\sigma_{ij} + \frac{2}{3}\theta\sigma_{ij} + \sigma_{ik}\sigma^k_j - \frac{1}{3}\sigma_{mn}\sigma^{mn}\delta^K_{ij}\right] - \varepsilon_{ij}$$

Dynamical to PDF Evolution

- **PDF evolution**
  - **fine-grained PDF**

$$\mathcal{P}_L^f(\mathbf{\Psi};\tau) = \delta_D\left[\psi(\tau) - \mathbf{\Psi}\right] = \delta_D\left[\delta_\rho(\tau) - \Delta_\rho\right]\delta_D\left[\boldsymbol{A}(\tau) - \boldsymbol{\mathcal{A}}\right],$$

  - **whose ensemble average relates to**

$$\langle P_L^f(\mathbf{\Psi};\tau)\rangle_L = \int d\mathbf{\Psi}' P_L(\mathbf{\Psi}';\tau)\delta_D(\mathbf{\Psi}' - \mathbf{\Psi}) = P_L(\mathbf{\Psi};\tau).$$

  - **Simply taking time derivatives**

$$\begin{aligned}\frac{\partial}{\partial\tau}\mathcal{P}_L(\mathbf{\Psi};\tau) &= \left\langle \frac{\partial}{\partial\tau}\mathcal{P}_L^f(\mathbf{\Psi};\tau)\right\rangle_L = \left\langle \frac{d\psi_\alpha}{d\tau}\left[\frac{\partial}{\partial\psi_\alpha}\delta_D(\psi(\tau) - \mathbf{\Psi})\right]\right\rangle_L \\ &= -\left\langle \chi_\alpha\left[\frac{\partial}{\partial\Psi_\alpha}P_L^f(\mathbf{\Psi};\tau)\right]\right\rangle_L = -\frac{\partial}{\partial\Psi_\alpha}\left\langle \chi_\alpha\mathcal{P}_L^f(\mathbf{\Psi};\tau)\right\rangle_L.\end{aligned}$$

Dynamical to PDF Evolution

- **PDF evolution**
  - Expressing as conditional average

$$\langle \chi_\alpha \mathcal{P}_L^f(\Psi;\tau)\rangle_L = \int d\Psi' d\boldsymbol{X}'\; X'_\alpha \delta_D(\Psi' - \Psi)\mathcal{P}_L(\Psi', \boldsymbol{X}';\tau)$$
$$= \int d\boldsymbol{X}'\; X'_\alpha \mathcal{P}_L(\boldsymbol{X}'|\Psi;\tau)\mathcal{P}_L(\Psi;\tau) = \langle \chi_\alpha|\Psi;\tau\rangle_L\, \mathcal{P}_L(\Psi;\tau)$$

  - Finally, we have a conservation equation

$$\frac{\partial}{\partial\tau}\mathcal{P}_L(\Psi;\tau) + \frac{\partial}{\partial\Psi_\alpha}\left[\langle \chi_\alpha|\Psi;\tau\rangle_L\, \mathcal{P}_L(\Psi;\tau)\right] = 0$$

- **Kinetic theory**
  - Considering $\psi = \{\mathbf{x}(\tau), \mathbf{p}(\tau)\}$
  - Single particle phase space evolution (Vlasov eq)

Dynamical to PDF Evolution

- **PDF evolution (*linear PDE*)**

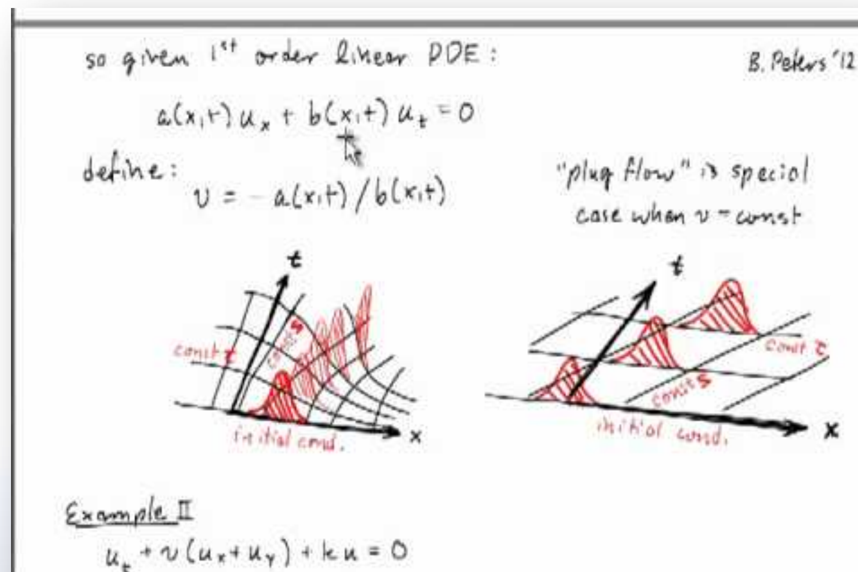
$$\frac{\partial}{\partial \tau}\mathcal{P}_L(\boldsymbol{\Psi};\tau) + \frac{\partial}{\partial \Psi_\alpha}\left[\langle \chi_\alpha | \boldsymbol{\Psi};\tau\rangle_L \, \mathcal{P}_L(\boldsymbol{\Psi};\tau)\right] = 0,$$

- **Method of Characteristics**
  - **PDE -> ODE**
  - **Textbook example**

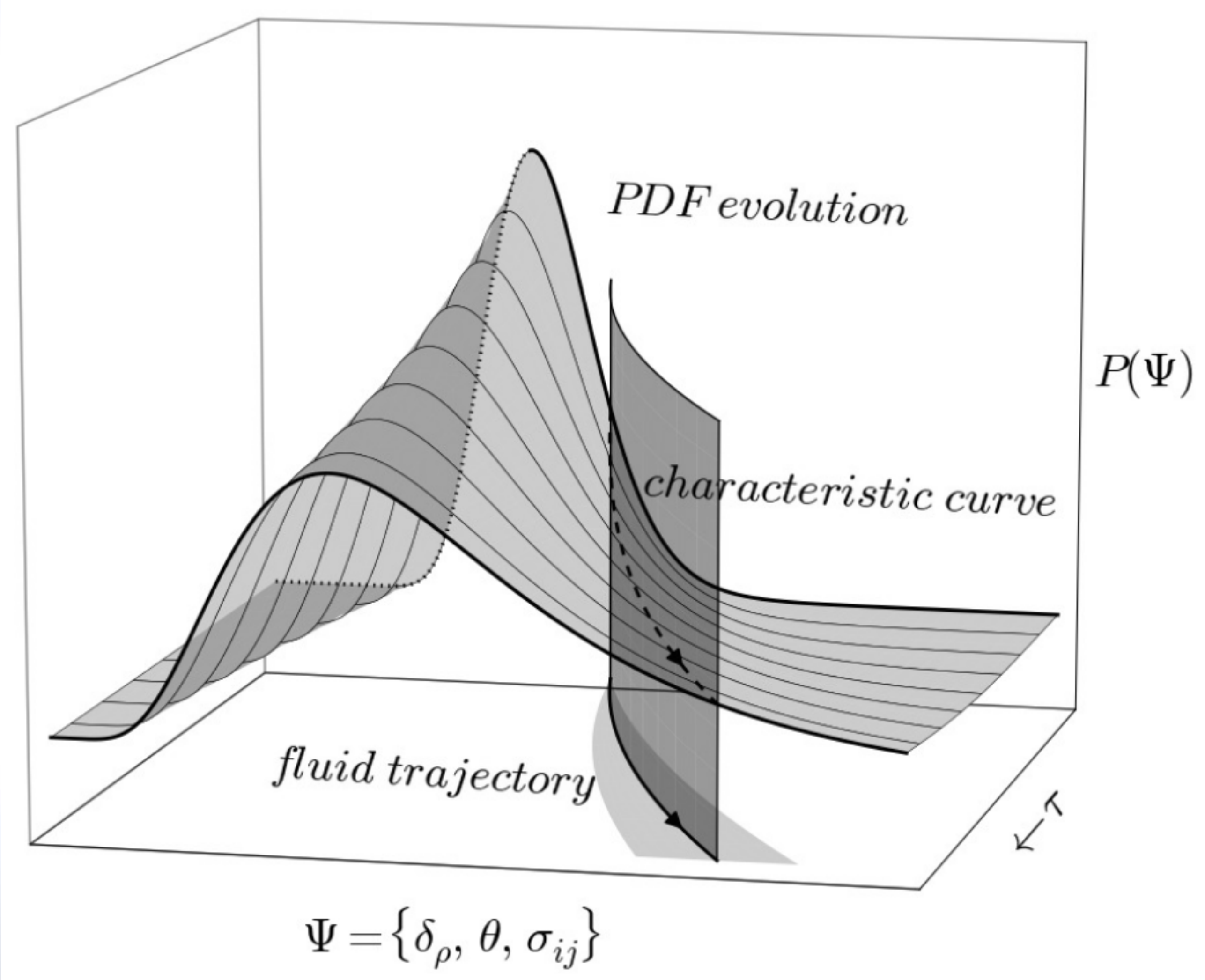
$$a(x,y)u_x + b(x,y)u_y = c(x,y)$$

  - **Integral curves**

$$\frac{dx}{ds} = a(x(s),y(s))$$
$$\frac{dy}{ds} = b(x(s),y(s))$$
$$\frac{dz}{ds} = c(x(s),y(s))$$



PDF Evolution & Characteristic Curve



Dynamical to PDF Evolution

- **From statistics**

$$\frac{\partial}{\partial \tau}\mathcal{P}_L(\mathbf{\Psi};\tau) + \frac{\partial}{\partial \Psi_\alpha}\left[\langle \chi_\alpha | \mathbf{\Psi};\tau\rangle_L \mathcal{P}_L(\mathbf{\Psi};\tau)\right] = 0,$$

- **Projected characteristic curve**

$$\frac{d}{d\tau}\mathbf{\Psi}(\tau) = \langle \boldsymbol{\chi} | \mathbf{\Psi};\tau\rangle_L.$$

- **Original dynamics** $\qquad \frac{d}{d\tau}\psi(\tau) = \boldsymbol{\chi}[\psi, \varepsilon_{ij}]$

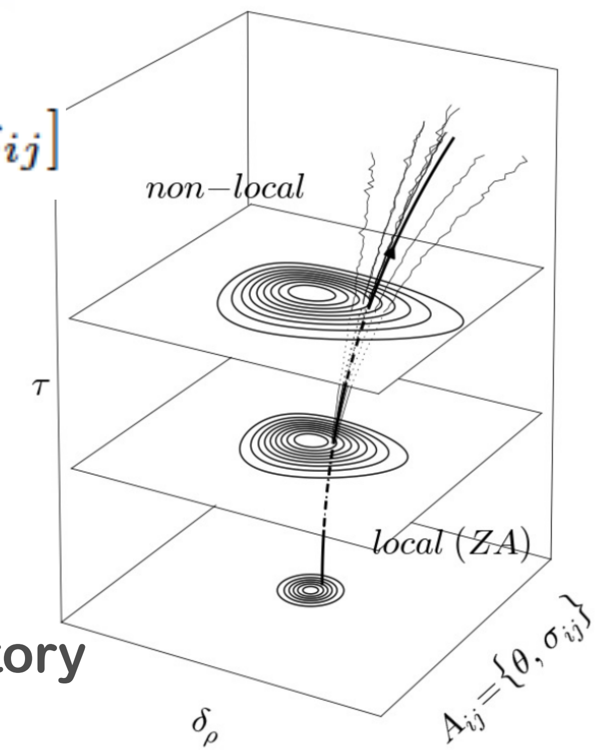
  – The only non-trivial equation:

$$\frac{d}{d\tau}\Sigma_{ij} + \mathcal{H}(\tau)\Sigma_{ij} + \frac{2}{3}\Theta\Sigma_{ij} + \Sigma_{ik}\Sigma_j^k - \frac{1}{3}\Sigma_{mn}\Sigma^{mn}\delta_{ij}^K = -\langle \varepsilon_{ij} | \mathbf{\Psi};\tau\rangle_L.$$

Projected Characteristic Curve

- **Mean evolution for** $\frac{d}{d\tau}\psi(\tau) = \boldsymbol{\chi}[\psi, \varepsilon_{ij}]$

$$\frac{d}{d\tau}\boldsymbol{\Psi}(\tau) = \langle \boldsymbol{\chi} | \boldsymbol{\Psi}; \tau \rangle_L.$$



- **Conserving one-point PDF**
  - **Statistical equivalence**
  - **repeating derivation with mean trajectory**
  - **Identical PDF evolution equation**

$$\frac{\partial}{\partial\tau}\mathcal{P}_L(\boldsymbol{\Psi};\tau) = \left\langle \frac{\partial}{\partial\tau}\mathcal{P}_L^f(\boldsymbol{\Psi};\tau) \right\rangle_L = \left\langle \frac{d\psi_\alpha}{d\tau} \left[\frac{\partial}{\partial\psi_\alpha}\delta_D(\psi(\tau) - \boldsymbol{\Psi}) \right] \right\rangle_L$$

$$= -\left\langle \chi_\alpha \left[\frac{\partial}{\partial\Psi_\alpha} P_L^f(\boldsymbol{\Psi};\tau) \right] \right\rangle_L = -\frac{\partial}{\partial\Psi_\alpha} \left\langle \chi_\alpha \mathcal{P}_L^f(\boldsymbol{\Psi};\tau) \right\rangle_L.$$

Statistics Evolution

- Lagrangian Evolution in Eulerian space
  - Tidal tensor should be averaged in Eulerian space
- define density weighted PDF

$$\mathcal{D}(\Psi;\tau) = (1+\Delta_\rho(\tau))\mathcal{P}_E(\Psi;\tau)$$

- And consider

$$\left\langle (1+\delta_\rho)\frac{d}{d\tau}Q(\psi^{tot}) \right\rangle_E$$

- We obtain

$$\frac{\partial}{\partial\tau}\mathcal{D}(\Psi;\tau) + \frac{\partial}{\partial\Psi_\alpha}\Big[\langle\chi_\alpha|\Psi;\tau\rangle_E\mathcal{D}(\Psi;\tau)\Big] = 0$$

- and

$$\langle\chi_\alpha|\Psi;\tau\rangle_L = \frac{1}{(1+\Delta_\rho)\mathcal{P}_E(\Psi)}\int dX\ X_\alpha(1+\Delta_\rho)\mathcal{P}_E(\boldsymbol{X},\Psi) = \langle\chi_\alpha|\Psi;\tau\rangle_E$$

Statistical Closure of Tidal Tensor

- Conditional average of tidal tensor, since

$$\frac{d}{d\tau}\Sigma_{ij} + \mathcal{H}(\tau)\Sigma_{ij} + \frac{2}{3}\Theta\Sigma_{ij} + \Sigma_{ik}\Sigma^{k}_{j} - \frac{1}{3}\Sigma_{mn}\Sigma^{mn}\delta^{K}_{ij} = -\langle \varepsilon_{ij} | \mathbf{\Psi}; \tau \rangle$$

- By definition,

$$\langle \varepsilon_{ij} | \mathbf{\Psi}; \tau \rangle \mathcal{P}(\mathbf{\Psi}; \tau) = \int d\boldsymbol{E}\; E_{ij}\mathcal{P}(\boldsymbol{E}, \mathbf{\Psi}; \tau) = \int d\boldsymbol{E}\; E_{ij}\mathcal{P}(\mathbf{\Gamma}; \tau)$$

- Gaussian variables

$$\langle \varepsilon_{ij} | \mathbf{\Psi}; \tau \rangle = \xi^{\varepsilon\psi}_{ij,\alpha} \left(\xi^{\psi\psi}\right)^{-1}_{\alpha\beta} \Psi_{\beta},$$

$$\begin{aligned} \langle \varepsilon_{ij} | \mathbf{\Psi}; \tau \rangle &= \frac{4\pi G_N \bar{\rho} a^2}{45} \left(\frac{15\sigma^2_{\delta\theta}}{2\sigma^2_{\theta\theta}}\right) \left(3\delta^K_{im}\delta^K_{jn} + 3\delta^K_{in}\delta^K_{jm} - 2\delta^K_{ij}\delta^K_{mn}\right) \mathcal{A}_{mn} \\ &= 4\pi G_N \bar{\rho} a^2 \left(\frac{\sigma^2_{\delta\theta}}{\sigma^2_{\theta\theta}}\right) \left(\mathcal{A}_{ij} - \frac{\Theta}{3}\delta^K_{ij}\right) = 4\pi G_N \bar{\rho} a^2 \left(\frac{\sigma^2_{\delta\theta}}{\sigma^2_{\theta\theta}}\right) \Sigma_{ij} \end{aligned}$$

Gaussian Closure

- For Gaussian closure, assume $\theta = -\mathcal{H}(\tau) f(\tau) \delta_\rho$

$$\sigma^2_{\delta\theta}/\sigma^2_{\theta\theta} = -1/\mathcal{H}(\tau) f(\tau)$$

- so that

$$\langle \varepsilon_{ij} | \mathbf{\Psi}; \tau \rangle = 4\pi G_N \bar{\rho} a^2 \left(\frac{\sigma^2_{\delta\theta}}{\sigma^2_{\theta\theta}} \right) \sigma_{ij} = \frac{-4\pi G_N \bar{\rho} a^2}{\mathcal{H} f} \sigma_{ij}$$

- Zel'dovich Approximation (ZA)
  - Since in ZA $\varepsilon_{ij} = -\frac{4\pi G_N a^2 \bar{\rho}}{\mathcal{H} f} \sigma_{ij}$, it's also local.
  - Statistical closure = ***Localization procedure***
  - So it is required that **Gaussian closure = ZA**

- Verified the method !

Gaussian Closure

- For Ga $-\mathcal{H}(\tau)f(\tau)\delta_\rho$

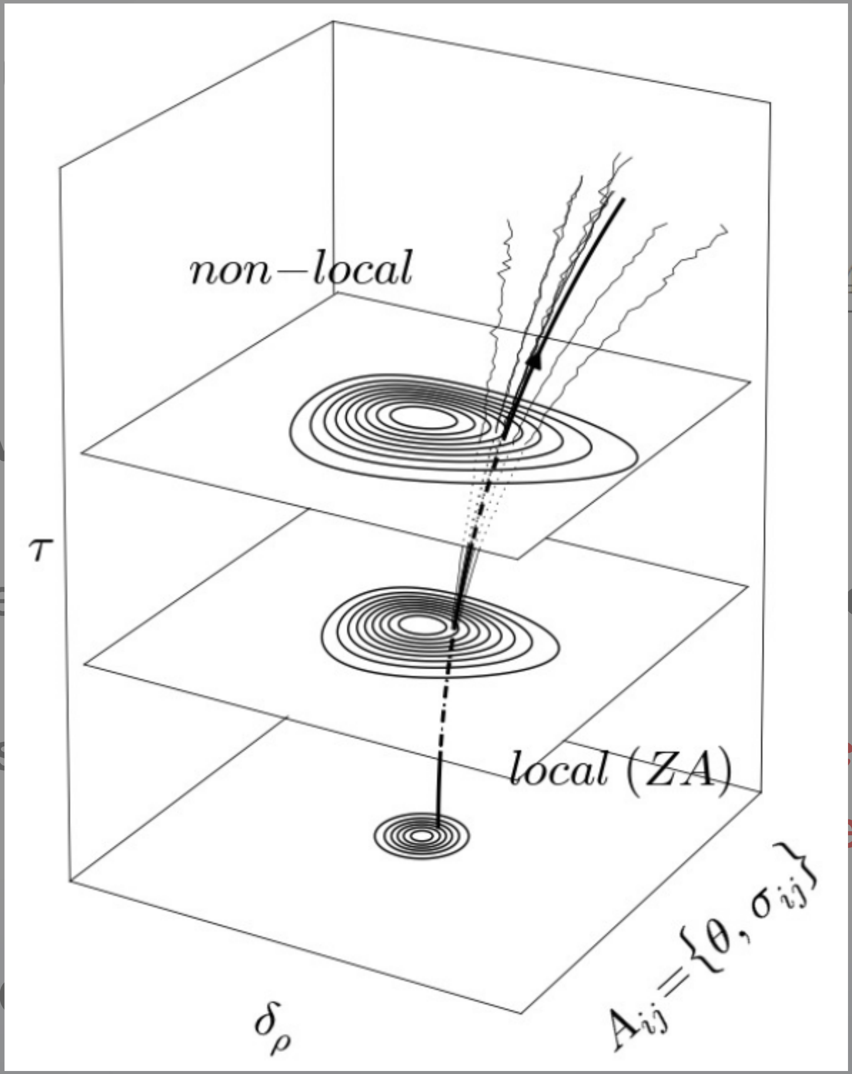
- so that $\frac{4\pi G_N\bar{\rho}a^2}{\mathcal{H}f}\sigma_{ij}$

- Zel'dov

  - Since o local.

  - Statis *cedure*
  - So it e = ZA

- Verifie



Weakly Non-Gaussian Closure

- Non-linearity & non-Guassianity
  - statistical closure: ***keep all nonlinearity***

$$\frac{d}{d\tau}\mathbf{\Psi}(\tau) = \langle \chi | \mathbf{\Psi}; \tau \rangle_L.$$

  - Except for the tidal tensor ...
  - By definition

$$\langle \varepsilon_{ij} | \mathbf{\Psi}; \tau \rangle \mathcal{P}(\mathbf{\Psi}; \tau) = \int d\boldsymbol{E}\; E_{ij} \mathcal{P}(\boldsymbol{E}, \mathbf{\Psi}; \tau) = \int d\boldsymbol{E}\; E_{ij} \mathcal{P}(\mathbf{\Gamma}; \tau)$$

  - In general, PDF expansion

$$\mathcal{P}(\mathbf{\Gamma}) = \exp\left[\sum_{n\geq 3} \frac{(-1)^n}{n!} \xi^{(n)}_{\alpha_1 \cdots \alpha_n} \frac{\partial^n}{\partial \Gamma_{\alpha_1} \cdots \partial \Gamma_{\alpha_n}}\right] \mathcal{P}_G(\mathbf{\Gamma})$$

Weakly Non-Gaussian Closure

- **Eventually the next order correction**

$$\Delta_{\langle \varepsilon_{ij}|\boldsymbol{\Psi}\rangle} = (Q_\rho \Delta_\rho + Q_\theta \Theta)\,\Sigma_{ij} + Q_{\Sigma^2} (\widetilde{\Sigma^2})_{ij},$$

  - **Where** $(\widetilde{\Sigma^2})_{ij} = \Sigma_i{}^m \Sigma_{mj} - \frac{1}{3}(\Sigma^{mn}\Sigma_{mn})\delta^K_{ij}$
  - **with coefficients**

$$Q_\rho = D_2 D_3 \left[\frac{1}{5}\tilde{\xi}_1^{(\varepsilon-\sigma)AA} + 6\left(\frac{D_1}{D_2}\right)\xi^{(\varepsilon-\sigma)\delta A}\right]$$

$$Q_\theta = D_3 D_5 \left[\frac{1}{5}\tilde{\xi}_1^{(\varepsilon-\sigma)AA} + 6\left(\frac{D_2}{D_5}\right)\xi^{(\varepsilon-\sigma)\delta A}\right]$$

$$Q_{\Sigma^2} = 4 D_3^2 \xi_4^{(\varepsilon-\sigma)AA}$$

- **In general:** $\langle \varepsilon_{ij}|\boldsymbol{\Psi};\,\tau\rangle = \mathscr{F}_{ij}(\boldsymbol{\Psi};\,\tau)$

Stochasticity

- **Stochastic noise ?** ***(extra dofs) e.g.*** $\varsigma_i(\mathbf{x}) = \frac{1}{\rho}\nabla_j(\rho\pi_{ij})$

$$\frac{d}{d\tau}\psi_\alpha(\tau) = \chi_\alpha(\boldsymbol{\psi},\tau) + \zeta_\alpha(\tau)$$

  - **Where** $\langle\zeta_\alpha(\mathbf{x},\tau)\zeta_\beta(\mathbf{x},\tau')\rangle = \xi^\zeta_{\alpha\beta}(\tau)\delta_D(\tau-\tau')$

- **PDF evolution: Fokker-Planck eq.**

$$\frac{\partial}{\partial\tau}\mathcal{D} + \frac{\partial}{\partial x_i}U_i\mathcal{D} + \frac{\partial}{\partial\Psi_\alpha}\langle\chi_\alpha|\boldsymbol{\Psi}\rangle\mathcal{D} = \frac{1}{2}\xi^\zeta_{\alpha\beta}(\tau)\frac{\partial^2}{\partial\Psi_\alpha\partial\Psi_\beta}\mathcal{D}$$

- **Langevin equation**

$$\frac{d}{d\tau}\Psi_\alpha(\tau) = \langle\chi_\alpha(\tau)|\boldsymbol{\Psi}(\tau)\rangle + \zeta_\alpha(\tau)$$

  - **In particular ?**

$$\frac{d}{d\tau}\omega_i + \mathcal{H}(\tau)\omega_i + \frac{2}{3}\theta\omega_i - \sigma_i^j\omega_j = (\text{stochastic terms})$$