New dual-conformally invariant off-shell integrals

Dung Nguyen, Marcus Spradlin, and Anastasia Volovich

Department of Physics
Brown University, Providence, Rhode Island 02912 USA

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(arXiv:0709.4665)
Rencontres de Moriond, QCD 2008

Outline

The emergence of a new symmetry

- There are evidences at weak coupling (up to 3 loops) and at strong coupling that the planar four-particle scattering amplitude may take the simple form:

$$\log(\mathcal{A}/\mathcal{A}_{\text{tree}}) = (\text{IR divergent terms}) + \frac{f(\lambda)}{8}\log^2(t/s) + c(\lambda) + \cdots$$

- At weak coupling, in $\mathcal{N} = 4$ SYM, so far up to 5 loops only dual-conformal invariant diagrams do contribute to the amplitude.
- At strong coupling, Alday-Maldacena prescription, involving T-dualizing AdS_5 space, makes dual-conformal symmetry manifest.

(Anastasiou, Bern, Dixon, Kosower'03; Bern, Dixon, Smirnov'05; Alday-Maldacena '07; Korchemsky et al. '07...)

Why do we use off-shell integrals?

- Off-shell: Typically finite in 4 dimensions
  On-shell : Typically infrared divergent as ϵ^{-2L}
- Off-shell: Typically one-term Mellin-Barnes representation
  On-shell : Typically thousands or tens of thousands terms (at 4 loops)
- Off-shell: known, simple analytic result for L-loop ladder diagram
  On-shell : Only at one loop, may not be possible at higher loop.
  (Usyukina and Davydychev '93, Broadhurst '93)
- "Magic identities" and relations among off-shell diagrams
  (Drummond, Henn, Smirnov and Sokatchev '07)

Here we are interested in the amplitudes of non-Abelian gauge bosons, relaxing $k_i^2 = 0$ would be fine. However, we don't know:

How to define off-shell scattering amplitude?
Which integrals do contribute to the amplitude?

Solving this problem, we can have several applications:

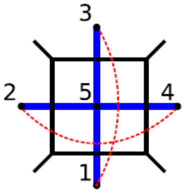
Calculation of the Cusp Anomalous Dimension

$$\log(\mathcal{A}/\mathcal{A}_{\mathrm{tree}}) = -\frac{\mathrm{f}(\lambda)}{8} \log^2(\mu^4/\mathrm{s\ t}) + \text{less singular terms}$$

The unexpected duality between light-like Wilson loops and scattering amplitudes

(Maldacena et al.'07, Korchemsky et al.'07, Brandhuber et al.'07)

An example of how to obtain dual conformal graphs



$$\mathcal{I}^{(1)}(k_1,k_2,k_3,k_4)=\int\frac{d^4p_1}{i\pi^2}\frac{(k_1+k_2)^2(k_2+k_3)^2}{p_1^2(p_1-k_1)^2(p_1-k_1-k_2)^2(p_1+k_4)^2}.$$

$$\mathcal{I}^{(1)}(x_1,x_2,x_3,x_4)=\int\frac{d^4x_5}{i\pi^2}\frac{x_{13}^2x_{24}^2}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2}.$$

where

$k_1=x_{12}, k_2=x_{23}, k_3=x_{34}, k_4=x_{41}, x_{ij}=x_i-x_j, p_1=x_{15}$.

Satisfying transformation $x_i^\mu=\frac{x_i^\mu}{x_i^2}$ $\Rightarrow$ Dual conformally invariant

Some properties of dual conformal diagrams

- Divergent *diagrams* would require a regulator, which will break the dual conformal symmetry, hence cannot be considered as dual conformal *integrals*.
- An integral with a numerator factor of $k_i^2 = \mu^2$ is *absent* when working on-shell, but does not necessarily vanish if first calculated for finite μ^2 then take $\mu^2 \to 0$.
- Four-point dual conformal integrals are constrained to be a function only of $x = \frac{\mu^4}{st}$.
- Degenerate integral (i.e. two or more of external momenta enter at the same vertex) must evaluate to a constant.

Algorithm

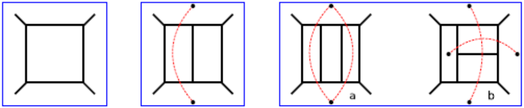
- We use QGRAF to generate all planar scalar four-point topologies with no tadpoles or bubbles or triangles.
- There are (1,1,4,25) distinct 1PI topologies at (1,2,3,4) loops.
- Adding numerator factors to make each diagram dual conformal, excluding "trivial" diagrams that are related to others.
- Another possible algorithm is to first use the result of *Schroder et al.* '02 to generate all planar 1PI vacuum graphs then attaching legs so that no triangles or bubbles remain. But at higher loops, extensive care must be taken.

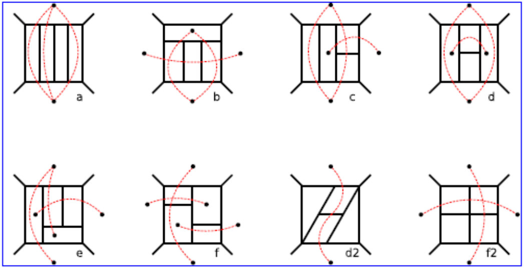
Results

	Finite	Divergent
Without μ^2 factors	Type I	Type II
With μ^2 factors	Type III	Type IV

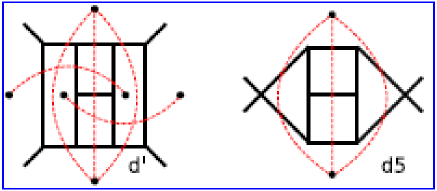
L	I	II	III	IV
1	1	0	0	0
2	1	0	0	0
3	2	0	2	0
4	8	2	9	9

Type I diagrams through 4 loops- a factor of st is suppressed

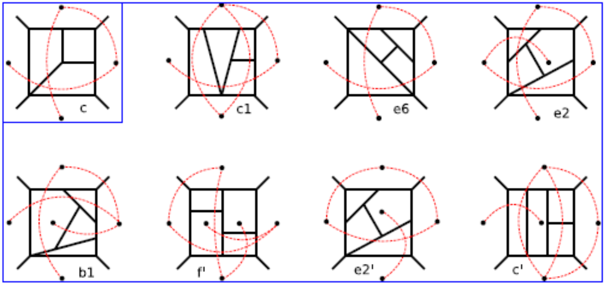


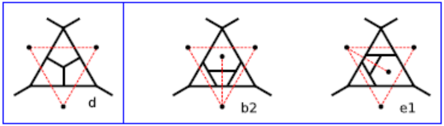


Type II diagrams through 4 loops

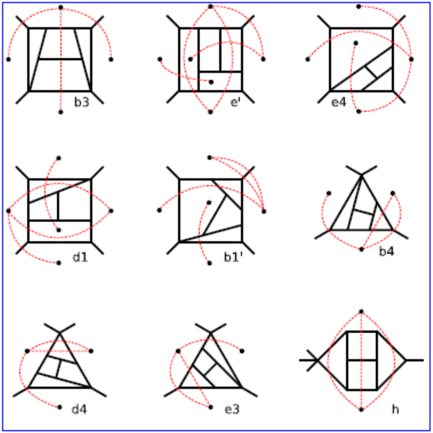


Type III diagrams through 4 loops





Type IV diagrams through 4 loops



Previously known integrals

$$\mathcal{I}^{(L)}(x) = \frac{2}{\sqrt{1-4x}} \left[\frac{(2L)!}{L!^2} \mathrm{Li}_{2L}(-y) + \sum_{\substack{k,l=0 \\ k+l \text{ even}}}^{L} \frac{(k+l)!(1-2^{1-k-l})}{k!l!(L-k)!(L-l)!} \zeta(k+l) \log^{2L-l-k} y \right]$$

where

$$y = \frac{2x}{1-2x+\sqrt{1-4x}}.$$

$$\mathcal{I}^{(3)a} = \mathcal{I}^{(3)b} = \mathcal{I}^{(3)}$$

$$\mathcal{I}^{(4)a} = \mathcal{I}^{(4)b} = \mathcal{I}^{(4)c} = \mathcal{I}^{(4)d} = \mathcal{I}^{(4)e} = \mathcal{I}^{(4)}.$$

Usyukina and Davydychev ’93, Drummond, Henn, Smirnov and Sokatchev ’07

3 loops - Diagram c

$$\mathcal{I}^{(3)c} = -\int \frac{d^5 z}{(2\pi i)^5} x^{z_2} \Gamma(-z_1)\Gamma(-z_2)\Gamma(z_2+1)\Gamma(-z_2+z_3+1)\Gamma(z_1+z_3-z_4+1)$$
$$\Gamma(-z_4)^2\Gamma(z_4-z_3)\Gamma(-z_2-z_5)^2\Gamma(z_1-z_4-z_5+1)$$
$$\Gamma(z_1-z_2+z_3-z_4-z_5+1)\Gamma(-z_1+z_4+z_5)\Gamma(z_2+z_4+z_5+1)$$
$$\Gamma(-z_1+z_2+z_4+z_5)\Gamma(-z_1-z_3+z_4+z_5-1)/(\Gamma(1-z_4)$$
$$\Gamma(-z_2-z_5+1)\Gamma(z_1-z_2+z_3-z_4-z_5+2)\Gamma(-z_1-z_2+z_4+z_5)$$
$$\Gamma(-z_1+z_2+z_4+z_5+1))$$

3 loops - Diagram d

$$\mathcal{I}^{(3)d} = \int \frac{d^4 z}{(2\pi i)^z} \Gamma(-z_1)\Gamma(z_1+1)\Gamma(-z_2)\Gamma(z_1+z_2-z_3+1)\Gamma(-z_3)^2\Gamma(z_3-z_1)$$
$$\Gamma(z_2-z_3-z_4+1)\Gamma(z_1+z_2-z_3-z_4+1)\Gamma(-z_4)^2\Gamma(z_3+z_4+1)$$
$$\Gamma(-z_2+z_3+z_4)\Gamma(-z_1-z_2+z_3+z_4)/(\Gamma(1-z_1)\Gamma(1-z_3)\Gamma(1-z_4)$$
$$\Gamma(z_1+z_2-z_3-z_4+2)\Gamma(-z_2+z_3+z_4+1)).$$

$$\mathcal{I}^{(3)d} \approx 20.73855510$$

4 loops - Diagram b 2

$$\mathcal{I}^{(4)b2} = \mathcal{I}^{(4)e1} = \int \frac{d^6 z}{(2\pi i)^6} \Gamma(-z_1)\Gamma(-z_2)\Gamma(-z_3)\Gamma(z_3+1)\Gamma(-z_1-z_2-z_4-2)$$
$$\Gamma(-z_1-z_2-z_3-z_4-2)\Gamma(-z_4)$$
$$\Gamma(z_1+z_2+z_4+3)\Gamma(z_1+z_2+z_3+z_4+3)\Gamma(z_1+z_3-z_5+1)\Gamma(-z_5)$$
$$\Gamma(z_5-z_3)\Gamma(z_2+z_4+z_5+2)\Gamma(-z_4-z_5-z_6-1)\Gamma(-z_6)^2$$
$$\Gamma(z_4+z_6+1)^2\Gamma(z_2+z_4+z_5+z_6+2)/(\Gamma(1-z_3)$$
$$\Gamma(-z_1-z_2-z_4-1)\Gamma(z_1+z_2+z_3+z_4+4)\Gamma(1-z_5)$$
$$\Gamma(z_2+z_4+z_5+3)\Gamma(1-z_6)\Gamma(z_4+z_6+2))$$

$$\mathcal{I}^{(4)b2} = \mathcal{I}^{(4)e1} = 70.59,$$

4 loops - Diagram c 1

$$\begin{aligned}\mathcal{I}^{(4)c1} = -\mathcal{I}^{(4)d2} = -\int \frac{d^7 z}{(2\pi i)^7}\, x^{z_2}\, &\Gamma(-z_1-1)\Gamma(z_1+2)\Gamma(-z_2)\Gamma(z_2+1)\\ &\Gamma(-z_1-z_3-1)\Gamma(-z_3)\Gamma(z_3-z_2)\Gamma(z_1-z_2+z_3+1)\Gamma(-z_4)\\ &\Gamma(z_1-z_2+z_3+z_5+2)\Gamma(-z_6)^2\Gamma(z_4+z_5-z_7+1)\\ &\Gamma(-z_1+z_2+z_4-z_6-z_7)\Gamma(z_4+z_5-z_6-z_7+1)\Gamma(-z_7)^2\Gamma(z_7-z_5)\\ &\Gamma(z_6+z_7+1)\Gamma(-z_4+z_6+z_7)\Gamma(-z_3-z_4-z_5+z_6+z_7-1)/\\ &(\Gamma(-z_1-z_2-1)\Gamma(1-z_3)\Gamma(z_1-z_2+z_3+2)\Gamma(1-z_6)\Gamma(1-z_7)\\ &\Gamma(z_4+z_5-z_6-z_7+2)\Gamma(-z_4+z_6+z_7+1))\end{aligned}$$

4 loops - Diagram e 2

$$\mathcal{I}^{(4)e2} = \mathcal{I}^{(4)b1} = -\int \frac{d^7 z}{(2\pi i)^7} x^{z_3} \Gamma(-z_1)\Gamma(-z_3)\Gamma(z_3+1)\Gamma(z_3-z_2)\Gamma(-z_4)^2$$
$$\Gamma(z_2-z_3+z_4+1)^2\Gamma(-z_5)^2\Gamma(z_1+z_2-z_5-z_6+2)\Gamma(-z_6)^2$$
$$\Gamma(z_5+z_6+1)\Gamma(-z_1-z_2+z_5+z_6-1)\Gamma(-z_1-z_2+z_3+z_5+z_6-1)$$
$$\Gamma(-z_4+z_5-z_7)\Gamma(-z_1-z_2-z_4+z_5+z_6-z_7-1)\Gamma(-z_7)$$
$$\Gamma(-z_3+z_4+z_7)\Gamma(z_1+z_2-z_3+z_4-z_5+z_7+2)$$
$$\Gamma(z_1-z_5-z_6+z_7+1)/(\Gamma(z_2-z_3+z_4-z_5+2)\Gamma(-z_4-z_6+1)$$
$$\Gamma(-z_1-z_2-z_3+z_5+z_6-1)\Gamma(-z_1-z_2+z_3+z_5+z_6)$$
$$\Gamma(-z_4-z_7+1)\Gamma(z_1+z_2-z_3+z_4-z_5-z_6+z_7+2))$$

4 loops - Diagram e 6

$$\begin{aligned}\mathcal{I}^{(4)e6} = -\int \frac{d^5 z}{(2\pi i)^5}\, x^{z_2}\, &\Gamma(-z_1)\Gamma(-z_2)^4\Gamma(z_2+1)^2\Gamma(-z_3)\Gamma(z_3+1)\\ &\Gamma(z_1+z_3-z_4+1)\Gamma(-z_4)^2\Gamma(z_4-z_3)\Gamma(z_1-z_4-z_5+1)\\ &\Gamma(z_1+z_3-z_4-z_5+1)\Gamma(-z_5)^2\Gamma(z_4+z_5+1)\Gamma(-z_1+z_4+z_5)\\ &\Gamma(-z_1-z_3+z_4+z_5)/(\Gamma(-2z_2)\Gamma(1-z_3)\Gamma(1-z_4)\Gamma(1-z_5)\\ &\Gamma(z_1+z_3-z_4-z_5+2)\Gamma(-z_1+z_4+z_5+1))\end{aligned}$$

4 loops - Diagram f 2

$$\mathcal{I}^{(4)f2} = \int \frac{d^{10}z}{(2\pi i)^{10}} x^{z_1} \Gamma(-z_1 - z_{10})^2 \Gamma(-z_2)\Gamma(-z_3)\Gamma(z_1 + z_{10} + z_3 + 1)^2 \Gamma(-z_4)$$
$$\Gamma(-z_5)\Gamma(-z_6)\Gamma(z_{10} + z_2 + z_6)\Gamma(-z_7)^2 \Gamma(z_4 + z_7 + 1)^2$$
$$\Gamma(-z_1 - z_4 - z_5 - z_8 - 2)(-z_1 - z_{10} - z_2 - z_4 - z_6 - z_7 - z_8 - 2)$$
$$\Gamma(-z_8)\Gamma(z_1 + z_8 + 2)(z_2 + z_4 + z_5 + z_7 + z_8 + 2)$$
$$\Gamma(-z_2 - z_3 - z_4 - z_5 - z_9 - 2)\Gamma(-z_{10} - z_2 - z_3 - z_6 - z_9)\Gamma(-z_9)$$
$$\Gamma(z_1 + z_{10} + z_2 + z_3 + z_6 + z_9 + 2)$$
$$\Gamma(z_{10} + z_2 + z_3 + z_4 + z_5 + z_6 + z_9 + 2)\Gamma(-z_1 + z_3 - z_8 + z_9)$$
$$\Gamma(z_2 + z_4 + z_5 + z_6 + z_8 + z_9 + 2)/(\Gamma(z_1 + z_{10} + z_3 + 2)\Gamma(-z_4 - z_5)$$
$$\Gamma(-z_1 - z_{10} - z_7)\Gamma(z_4 + z_7 + 2)\Gamma(-z_1 - z_{10} - z_2 - z_6 - z_8)$$
$$\Gamma(-z_3 - z_9)\Gamma(-z_1 + z_{10} + z_2 + z_3 + z_6 - z_8 + z_9)$$
$$\Gamma(z_1 + z_{10} + z_2 + z_3 + z_4 + z_5 + z_6 + z_8 + z_9 + 4))$$

Infrared singularity structure

$$\mathcal{I}^{(1)} = \log^2 x + \mathcal{O}(1)$$

$$\mathcal{I}^{(2)} = \frac{1}{4}\log^4 x + \frac{\pi^2}{2}\log^2 x + \mathcal{O}(1),$$

$$\mathcal{I}^{(3)} = \frac{1}{36}\log^6 x + \frac{5\pi^2}{36}\log^4 x + \frac{7\pi^4}{36}\log^2 x + \mathcal{O}(1),$$

$$\mathcal{I}^{(4)} = \frac{1}{576}\log^8 x + \frac{7\pi^2}{432}\log^6 x + \frac{49\pi^4}{864}\log^4 x + \frac{31\pi^6}{432}\log^2 x + \mathcal{O}(1).$$

Infrared singularity structure

$$\mathcal{I}^{(3)c} = \frac{\zeta(3)}{3}\log^3 x - \frac{\pi^4}{30}\log^2 x + 14.32388625 \log x + \mathcal{O}(1)$$

$$\begin{aligned}
\mathcal{I}^{(4)d2} = -\mathcal{I}^{(4)c1} &= -\frac{\zeta(3)}{12}\log^5 x + \frac{7\pi^4}{720}\log^4 x - 6.75193310 \log^3 x \\
&\quad + 15.45727322 \log^2 x - 41.26913 \log x + \mathcal{O}(1), \\
\mathcal{I}^{(4)f2} &= \frac{1}{144}\log^8 x + \frac{7\pi^2}{108}\log^6 x + \frac{149\pi^4}{1080}\log^4 x \\
&\quad + 64.34694867 \log^2 x + \mathcal{O}(1), \\
\mathcal{I}^{(4)e6} &= -20.73855510 \log^2 x + \mathcal{O}(1), \\
\mathcal{I}^{(4)e2} = \mathcal{I}^{(4)b1} &= -\frac{\pi^4}{720}\log^4 x + 1.72821293 \log^3 x \\
&\quad - 12.84395616 \log^2 x + 52.34900 \log x + \mathcal{O}(1)
\end{aligned}$$

Summary

- We have classified all four-point dual conformal diagrams through four loops.
- At (1,2,3,4) loops in addition to known (1,1,2,8) integrals we find (0,0,2,9) new ones.
- Of the total (1,1,4,17), in which (1,1,2,5) are known, we find Mellin-Barnes representation for (0,0,2,8) integrals and evaluate their infrared singularity structure explicitly, but left (0,0,0,4) integrals for future work ($I^{(4)f}, I^{(4)f'}, I^{(4)e2'}, I^{(4)c'}$).
- Outlook
  - Define a consistently general off-shell scattering amplitude?
  - Calculate the cusp anomalous dimension $f(\lambda)$?
  - Classify dual conformal diagrams at higher loops?
  - Checking the Wilson loops - Scattering amplitude Duality ?