Theory of the Lamb shift in muonic and electronic hydrogen for determination of the proton charge radius

Krzysztof Pachucki

University of Warsaw

QCD Moriond, March 25, 2011

Discrepancy between different determinations of the proton charge radius

- (electronic) hydrogen Lamb shift: $r_p = 0.880\,2(80)$ fm, [CODATA, 2008]
- unpolarized electron scattering: $r_p = 0.879(5)(4)(2)(4)$ fm, [Mainz, 2010]
- polarized electron scattering: $r_p = 0.875(10)$ fm [Jefferson Lab]
- muonic hydrogen Lamb shift: $r_p = 0.841\,84(67)$ fm, [PSI, 2020], 5 standard deviations from the CODATA value.

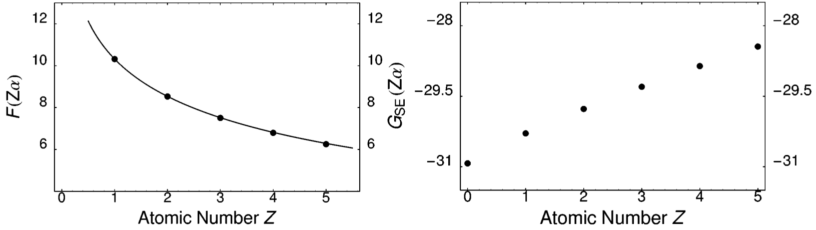
Contributions to the Lamb shift in *eH*

- one-loop electron self-energy and vacuum polarization
- two-loops
- three-loops
- pure recoil correction
- radiative recoil correction
- finite nuclear size, and polarizability

One-loop electron self-energy

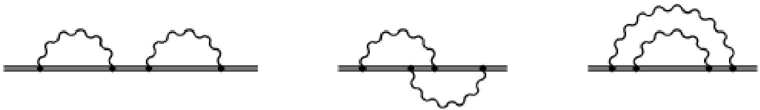
- $\delta_{SE} E = \frac{e^2}{(2\pi)^4 i} \int d^d k \frac{1}{k^2} \langle \bar{\psi} | \gamma^\mu \frac{1}{\not{p} - \not{k} + \gamma^0 Z\alpha/r - m} \gamma_\mu | \psi \rangle - \delta m \langle \bar{\psi} | \psi \rangle$
- Numerical evaluation of the one-loop self-energy,
  U. Jentschura (1999)

$$\delta E = \frac{\alpha}{\pi} (Z\alpha)^4 \, m \, F(Z\alpha)$$



$$\begin{aligned} F(Z\alpha) &= A_{40} + A_{41} \ln(Z\alpha)^{-2} + (Z\alpha) A_{50} \\ &\quad + (Z\alpha)^2 [A_{62} \ln^2(Z\alpha)^{-2} + A_{61} \ln(Z\alpha)^{-2} + G_{60}] \end{aligned}$$

Two-loop electron self-energy correction

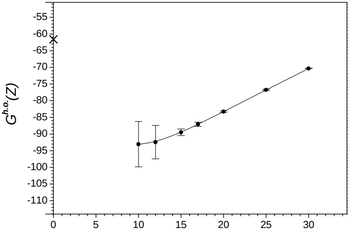


Electron propagators include external Coulomb field, external legs are bound state wave functions.

The expansion of the energy shift in powers of $Z\alpha$

$$\begin{aligned}\delta^{(2)}E &= m\left(\frac{\alpha}{\pi}\right)^2 F(Z\alpha)\\ F(Z\alpha) &= B_{40} + (Z\alpha)\,B_{50} + (Z\alpha)^2\Big\{[\ln(Z\alpha)^{-2}]^3\,B_{63}\\ &\quad + [\ln(Z\alpha)^{-2}]^2\,B_{62} + \ln(Z\alpha)^{-2}\,B_{61} + G(Z\alpha)\Big\}\end{aligned}$$

Direct numerical calculation versus analytical expansion



$G_{60}(1) \approx -86(15)$ (Yerokhin, 2009), uncertainty $\delta E(1S) = \pm 1.5$ kHz

$B_{60} = -61.6(9.2)$ (K.P., U.J., 2003)

uncertainty due to the unknown high energy contribution from the class of about 80 diagrams

discrepancy in the proton charge radius $\rightarrow \delta E(1S) = 94$ kHz

Recoil corrections

- finite nuclear mass effects, beyond the Dirac equation
- leading $O(\alpha^5)$ terms are known for an arbitrary mass ratio

$$\delta E^{(5)} = \frac{\mu^3}{m\,M}\,\frac{(Z\,\alpha)^5}{\pi\,n^3}\left\{\frac{1}{3}\,\delta_{l0}\,\ln(Z\,\alpha)^{-2} - \frac{8}{3}\,\ln k_0(n,l) - \frac{1}{9}\,\delta_{l0} - \frac{7}{3}\,a_n \right. \\ \left. -\frac{2}{M^2-m^2}\,\delta_{l0}\left[M^2\,\ln\frac{m}{\mu} - m^2\,\ln\frac{M}{\mu}\right]\right\}$$

where

$$a_n = -2\left[\ln\frac{2}{n} + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) + 1 - \frac{1}{2\,n}\right]\delta_{l0} + \frac{1-\delta_{l0}}{l\,(l+1)\,(2\,l+1)}$$

- many higher order corrections included

Nuclear size and polarizability corrections

- leading finite size

$$E_{FS} = \frac{2\,\pi\,Z\,\alpha}{3}\,\phi^2(0)\,r_p^2$$

- higher order finite size $\sim 10^{-5}$

$$\delta E_{FS} = -E_{FS}\,C_\eta\,\frac{r_p}{\lambda\!\!\!^{-}}\,Z\,\alpha$$

  where $C_\eta = 1.7(1)$

- finite size combined with SE and VP $\sim O(Z\,\alpha^2)$
- proton polarizability: $\delta_{\mathrm{pol}} E = -0.087(16)\,\frac{\delta_{l0}}{n^3}\,h\,kHz$

Experimental results for hydrogen and r_p

- $\nu_{exp.}(2P_{1/2} - 2S_{1/2}) = 1\,057\,845.0(9.0)$ kHz, [Lundeen, Pipkin, 1994]
- $\nu_{\rm exp}(1S_{1/2} - 2S_{1/2}) = 2\,466\,061\,413\,187.074(34)$ kHz, [MPQ, 2004]
- $\nu_{\rm exp}(2S_{1/2} - 8D_{5/2}) = 770\,859\,252\,849.5(5.9)$ kHz, [Paris, 2001]
- global fit to the hydrogen data $\Rightarrow$

$$r_p = 0.880\,2(80)\ \text{fm}$$

- proton charge radius from muonic hydrogen Lamb shift is calculated in a similar way

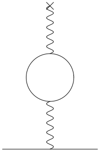
Theory of μH energy levels

- μH is essentially a nonrelativistic atomic system
- muon and proton are treated on the same footing
- $m_\mu / m_e = 206.768 \Rightarrow \beta = m_e/(\mu\,\alpha) = 0.737$ the ratio of the Bohr radius to the electron Compton wavelength
- the electron vacuum polarization dominates the Lamb shift

$$E_L = \int d^3r\, V_{vp}(r)\,(\rho_{2P} - \rho_{2S}) = 205.006\,\mathrm{meV}$$

- important corrections: second order, two-loop vacuum polarization, and the muon self-energy
- other corrections are much smaller than the discrepancy of 0.3 meV.

One-loop electron vacuum-polarization



- the electron loop modifies the Coulomb interaction by

$$V_{vp}(r) = -\frac{Z\,\alpha}{r}\,\frac{\alpha}{\pi}\int_4^\infty \frac{d(q^2)}{q^2}\,e^{-m_e\,q\,r}\,u(q^2)$$

$$u(q^2) = \frac{1}{3}\sqrt{1-\frac{4}{q^2}}\left(1+\frac{2}{q^2}\right)$$

$$E_L = \langle 2P|V_{vp}|2P\rangle - \langle 2S|V_{vp}|2S\rangle = 205.006\,\mathrm{meV}$$

Leading relativistic correction

No Dirac equation for the finite mass nucleus

$$\delta H = -\frac{p^4}{8\,m^3} - \frac{p^4}{8\,M^3} + \frac{\alpha}{r^3}\left(\frac{1}{4\,m^2} + \frac{1}{2\,m\,M}\right)\vec{r}\times\vec{p}\cdot\vec{\sigma} + \frac{\pi\,\alpha}{2}\left(\frac{1}{m^2} + \frac{1}{M^2}\right)\delta^3(r) - \frac{\alpha}{2\,m\,M\,r}\left(p^2 + \frac{\vec{r}\,(\vec{r}\vec{p})\,\vec{p}}{r^2}\right)$$

$$\delta E = \langle l, j, m_j|\delta H|l, j, m_j\rangle = \frac{(Z\,\alpha)^4\,\mu^3}{2\,n^3\,m_p^2}\left(\frac{1}{j+\frac{1}{2}} - \frac{1}{l+\frac{1}{2}}\right)(1-\delta_{l0})$$

$$\delta E_L = \frac{\alpha^4\mu^3}{48\,m_p^2} = 0.057\mathrm{meV}$$

valid for an arbitrary mass ratio

$O(\alpha^5)$ recoil correction

- pure recoil coorection

$$E(n,l) = \frac{\mu^3}{m_\mu\, m_p}\frac{(Z\,\alpha)^5}{\pi\, n^3}\left\{\frac{2}{3}\,\delta_{l0}\,\ln\left(\frac{1}{Z\,\alpha}\right) - \frac{8}{3}\,\ln k_0(n,l) - \frac{1}{9}\,\delta_{l0} - \frac{7}{3}\,a_n - \frac{2}{m_p^2 - m_\mu^2}\,\delta_{l0}\left[m_p^2\,\ln\left(\frac{m_\mu}{\mu}\right) - m_\mu^2\,\ln\left(\frac{m_p}{\mu}\right)\right]\right\}$$

$$a_n = -2\left(\ln\frac{2}{n} + (1 + \frac{1}{2} + \ldots + \frac{1}{n}) + 1 - \frac{1}{2\,n}\right)\delta_{l0} + \frac{1-\delta_{l0}}{l\,(l+1)\,(2\,l+1)}$$

It contributes the amount of

$$\delta E_L = -0.045\,\mathrm{meV}\,.$$

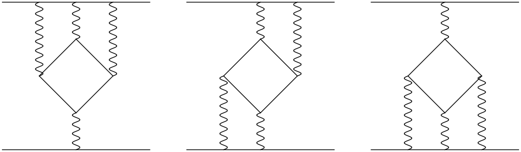
- proton self-energy, electromagnetic formfactors are infrared divergent

$$E(n,l) = \frac{4\,\mu^3\,(Z^2\,\alpha)\,(Z\,\alpha)^4}{3\,\pi\,n^3\,m_p^2}\left(\delta_{l0}\,\ln\left(\frac{m_p}{\mu\,(Z\,\alpha)^2}\right) - \ln k_0(n,l)\right)\,. \quad (1)$$

It contributes

$$\delta E_L = -0.010\,\mathrm{meV}\,. \quad (2)$$

Light by light diagrams



- $\delta E_L = -0.0009$ meV
- significant cancellation between diagrams
- S.G. Karshenboim *et al.*, arXiv:1005.4880

Proton finite size correction

- in muonic atoms nuclear finite size effects give a large contribution to energy levels:

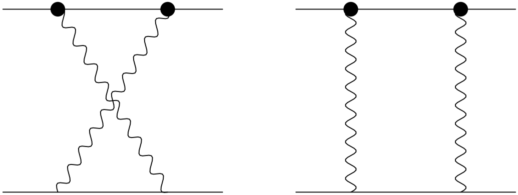
$$E_{FS} = \frac{2\,\pi\,\alpha}{3}\,\phi^2(0)\,\langle r_p^2\rangle\,\delta_{l0}$$

$$\phi^2(0) = \frac{(\mu\,\alpha)^3}{\pi}\,\delta_{l0}$$

- relativistic corrections to $\phi^2(0)$
- electron vp corrections to $\phi^2(0)$
- muon self-energy correction to $\phi^2(0)$
- in total: $E_{FS} = -5.2262\, r_p^2/\mathrm{fm}^2$ meV

Nuclear structure effects

- when nuclear excitation energy is much larger than the atomic energy, the two-photon exchange scattering amplitude gives the dominating correction



Two-photon scattering amplitude

$$\delta E = -\frac{e^4}{2}\,\phi^2(0)\int\frac{d^4q}{(2\,\pi)^4\,i}\,\frac{1}{q^4}\left[T^{\mu\nu}-t^{\mu\nu}(M)\right]t_{\mu\nu}(m)$$
$$= -2\,e^4\,\phi^2(0)\,\frac{m}{M}\int\frac{d^4q}{(2\,\pi)^4\,i}\,\frac{(T_2-t_2)(q^2-\nu^2)-(T_1-t_1)\,(q^2+2\,\nu^2)}{q^4\,(q^4-4\,m^2\nu^2)}$$

where for a point–like proton $T^{\mu\nu}\equiv t^{\mu\nu}(M)$ and

$$T^{\mu\nu} = -i\int d^4q\,e^{i\,q\,(x-x')}\,\langle P|T\,j^\mu(x)\,j^\nu(x')|P\rangle$$
$$= -\left(g^{\mu\nu}-\frac{q^\mu\,q^\nu}{q^2}\right)\frac{T_1}{M}+\left(t^\mu-\frac{\nu}{q^2}\,q^\mu\right)\left(t^\nu-\frac{\nu}{q^2}\,q^\nu\right)\frac{T_2}{M}\,,$$

$t=(1,0,0,0)$, $P=M\,t$ is a proton momentum at rest and $\nu=q^0$.

$\delta E=\delta E_{\rm el}+\delta E_{\rm inel}$

Elastic contribution

$$T_1^{pole} = -\frac{q^4}{(q^4 - 4\,M^2\,\nu^2)}\,(F_1 + F_2)^2$$

$$T_2^{pole} = \frac{1}{(q^4 - 4\,M^2\,\nu^2)}\,(4\,M^2\,q^2\,F_1^2 - q^4\,F_2^2)$$

$$\delta E_{el} = -2\,e^4\,\phi^2(0)\,\frac{m}{M}\int\frac{d^4q}{(2\,\pi)^4\,i}\,\frac{(T_2 - t_2)(q^2 - \nu^2) - (T_1 - t_1)\,(q^2 + 2\,\nu^2)}{q^4\,(q^4 - 4\,m^2\nu^2)}$$

$$= 0.031\,\mathrm{meV}\ \text{for}\ 2P - 2S\ \text{transition [Carlson } et\ al\text{, 2011]}$$

in the infinite nuclear mass limit

$$\delta E_{el} = -\frac{\mu^3}{\pi\,n^3}\delta_{l0}\,(Z\,\alpha)^5\int_0^\infty\frac{dq}{q^4}\,16\,m_\mu\,\left[G_E^2(-q^2) - 1 + 2\,G_E'(0)\,q^2\right]$$

$$= \frac{(Z\,\alpha)^5}{3\,n^3}\,\mu^4\,\delta_{l0}\,\langle r_p^3\rangle_{(2)}$$

$$\left(\langle r_p^3\rangle_{(2)}\right)^{1/3} = 1.394(22)\,\mathrm{fm} \Rightarrow \delta E_{el} = 0.021\,\mathrm{meV}$$

Inelastic and subtraction contributions

- dispersion relations

$$T_2(\nu, q^2) = T_2^{pole} - \int_{\nu_{th}^2}^{\infty} d\nu'^2 \, \frac{W_2(\nu', q^2)}{\nu'^2 - \nu^2}$$

$$T_1(\nu, q^2) = T_1^{pole} + T_1(0, q^2) - \nu^2 \int_{\nu_{th}^2}^{\infty} \frac{d\nu'^2}{\nu'^2} \, \frac{W_1(\nu', q^2)}{\nu'^2 - \nu^2}$$

$$\delta E = -2\, e^4\, \phi^2(0)\, \frac{m}{M} \int \frac{d^4 q}{(2\,\pi)^4\, i} \, \frac{T_2(q^2 - \nu^2) - T_1\,(q^2 + 2\,\nu^2)}{q^4\,(q^4 - 4\, m^2 \nu^2)}$$

- W_1 and W_2 are known from the inelastic scattering off the proton
- $\lim_{q^2 \to 0} \frac{T_1(0, q^2)}{q^2} = \frac{M}{\alpha}\, \beta_M$
- $\delta E_{inel} = 12.7(5)\ \mu$eV, [Carlson *et al.* 2011]
- the total proton structure contribution $\delta E_L = 36.9(2.4)\ \mu$eV is much too small to explain the discrepancy

Final results

- $\Delta E = E(2P_{3/2}(F=2)) - E(2S_{1/2}(F=1))$
- experimental result: $\Delta E = 206.2949(32)\ \text{meV}$
- total theoretical result from [U. Jenschura, 2011]

$$\Delta E = \left(209.9974(48) - 5.2262\frac{r_p^2}{\text{fm}^2}\right)\text{meV} \Rightarrow$$

- $r_p = 0.841\,69(66)\ \text{fm}$

Analysis of discrepancy

possible sources of discrepancy:

- mistake in the QED calculations ? μH checked by many and only few corrections contribute at the level of discrepancy
- vp verified by an agreement $\sim 10^{-6}$ for $3D-2P$ transition in ^{24}Mg and ^{28}Si
- large Zemach moment $(r_p^{(2)})^3$ ruled out by the low energy electron-proton scattering [Friar, Sick, 2005], [Cloët, Miller, 2010], [Distler, Bernauer, Walcher, 2010]
- nuclear structure correction ? much too small
- possible new light particles ? ruled out by muon $g-2$ and other low energy Standard Model tests [Barger, Cheng-Wei Chiang, Wai-Yee Keung, Marfatia, 2010]
- violation of the universality in the short distance $\sim$ 1fm lepton-proton interaction ? universality verified experimentally by a direct comparison of $e-p$ with $\mu-p$ scattering
- **5 σ shift in Rydberg constant**, [Randolf Pohl, 2010]