

ASYMPTOTIC HIGH-ENERGY BEHAVIOR OF HADRONIC TOTAL CROSS SECTIONS FROM LATTICE QCD

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We will show how a *universal* and *Froissart-like* (i.e., of the kind $B \log^2 s$) hadron-hadron total cross section can emerge in QCD asymptotically at high energy, finding indications for this behavior from the lattice. The functional integral approach provides the “natural” setting for achieving this result, since it encodes the energy dependence of hadronic scattering amplitudes in a single *elementary* object, i.e., a proper correlation function of two Wilson loops.

1 Introduction

Present-day experimental observations (up to a center-of-mass total energy $\sqrt{s} = 7$ TeV, reached at the LHC pp collider¹) seem to support the following asymptotic high-energy behavior of hadronic total cross sections: $\sigma_{\text{tot}}^{(hh)}(s) \sim B \log^2 s$, with a *universal* (i.e., *not* depending on the particular hadrons involved) coefficient $B \simeq 0.3$ mb.² This behavior is consistent with the well-known *Froissart-Lukaszuk-Martin* (FLM) *theorem*³, according to which, for $s \rightarrow \infty$, $\sigma_{\text{tot}}^{(hh)}(s) \leq (\pi/m_\pi^2) \log^2(s/s_0)$, where m_π is the pion mass and s_0 is an unspecified squared mass scale. As we believe QCD to be the fundamental theory of strong interactions, we also expect that it correctly predicts from first principles the behavior of hadronic total cross sections. However, in spite of all the efforts, a satisfactory solution to this problem is still lacking. (For some theoretical supports to the universality of B , see Ref.⁴ and references therein.)

This problem is part of the more general problem of high-energy elastic scattering at low transferred momentum, the so-called *soft high-energy scattering*. As *soft* high-energy processes possess two different energy scales, the total center-of-mass energy squared s and the transferred momentum squared t , smaller than the typical energy scale of strong interactions ($|t| \lesssim 1 \text{ GeV}^2 \ll s$), we cannot fully rely on perturbation theory (PT). A nonperturbative (NP) functional-integral approach in the framework of QCD has been proposed in Ref.⁵ and further developed in Ref.⁶ In this approach, for example, the elastic scattering amplitude $\mathcal{M}_{(hh)}$ of two *mesons*, of the same mass m for simplicity, can be reconstructed from the scattering amplitude $\mathcal{M}_{(dd)}$ of two dipoles of fixed transverse sizes $\vec{r}_{1,2\perp}$, and fixed longitudinal-momentum fractions $f_{1,2}$ of the quarks in the two dipoles, after folding with squared wave functions $\rho_{1,2} = |\psi_{1,2}|^2$ describing the interacting hadrons,⁶

$$\mathcal{M}_{(hh)}(s, t) = \int d^2\nu \rho_1(\nu_1) \rho_2(\nu_2) \mathcal{M}_{(dd)}(s, t; \nu_1, \nu_2) \equiv \langle \langle \mathcal{M}_{(dd)}(s, t; 1, 2) \rangle \rangle, \quad (1)$$

where $\nu_i = (\vec{r}_{i\perp}, f_i)$ denotes collectively the dipole variables, $d^2\nu = d\nu_1 d\nu_2$, $\int d\nu_i = \int d^2\vec{r}_{i\perp} \int_0^1 df_i$, and $\int d\nu_i \rho_i(\nu_i) = 1$. In turn, the dipole-dipole (dd) scattering amplitude is obtained from the

(properly normalized) correlation function (CF) of two Wilson loops (WL) in the fundamental representation, defined in Minkowski spacetime, running along the paths made up of the quark and antiquark classical straight-line trajectories, and thus forming a hyperbolic angle $\chi \simeq \log(s/m^2)$ in the longitudinal plane. The paths are cut at proper times $\pm T$ as an infrared regularization, and closed by straight-line “links” in the transverse plane, in order to ensure gauge invariance; eventually, $T \rightarrow \infty$. It has been shown in Refs.^{7,8,9} that the relevant Minkowskian CF $\mathcal{G}_M(\chi; T; \vec{z}_\perp; \nu_1, \nu_2)$ (\vec{z}_\perp being the *impact parameter*, i.e., the transverse separation between the two dipoles) can be reconstructed, by means of *analytic continuation*, from the Euclidean CF of two Euclidean WL, $\mathcal{G}_E(\theta; T; \vec{z}_\perp; \nu_1, \nu_2) \equiv \langle \mathcal{W}_1^{(T)} \mathcal{W}_2^{(T)} \rangle / (\langle \mathcal{W}_1^{(T)} \rangle \langle \mathcal{W}_2^{(T)} \rangle) - 1$, where $\langle \dots \rangle$ is the average in the sense of the Euclidean QCD functional integral. The Euclidean WL $\mathcal{W}_{1,2}^{(T)} = N_c^{-1} \text{Tr} \{ T \exp[-ig \oint_{\mathcal{C}_{1,2}} A_\mu(x) dx_\mu] \}$ are calculated on the following quark $[q]$ -antiquark $[\bar{q}]$ straight-line paths, $\mathcal{C}_i : X_i^{q[\bar{q}]}(\tau) = z_i + \frac{p_i}{m} \tau + f_i^{q[\bar{q}]} r_i$, with $\tau \in [-T, T]$, and closed by straight-line paths in the transverse plane at $\tau = \pm T$. Here $p_{1,2} = m(\pm \sin \frac{\theta}{2}, \vec{0}_\perp, \cos \frac{\theta}{2})$, θ being the angle formed by the two Euclidean trajectories (i.e., $p_1 \cdot p_2 = m^2 \cos \theta$), $r_i = (0, \vec{r}_{i\perp}, 0)$, $z_i = \delta_{i1}(0, \vec{z}_\perp, 0)$ and $f_i^q \equiv 1 - f_i$, $f_i^{\bar{q}} \equiv -f_i$. We define also the CFs with the infrared cutoff removed as $\mathcal{C}_{E,M} \equiv \lim_{T \rightarrow \infty} \mathcal{G}_{E,M}$. The dd scattering amplitude is then obtained from $\mathcal{C}_E(\theta; \dots)$ [with $\theta \in (0, \pi)$] by means of analytic continuation as ($t = -|\vec{q}_\perp|^2$)

$$\begin{aligned} \mathcal{M}_{(dd)}(s, t; \nu_1, \nu_2) &\equiv -i 2s \int d^2 \vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \mathcal{C}_M(\chi; \vec{z}_\perp; \nu_1, \nu_2) \\ &= -i 2s \int d^2 \vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \mathcal{C}_E(\theta \rightarrow -i\chi; \vec{z}_\perp; \nu_1, \nu_2). \end{aligned} \quad (2)$$

In Refs.^{10,11} the CF \mathcal{C}_E were calculated in *quenched* QCD by Monte Carlo simulations in *Lattice Gauge Theory* (LGT), at lattice spacing $a(\beta = 6) \simeq 0.1$ fm, on a 16^4 hypercubic lattice, using loops of transverse size a at angles $\cot \theta = 0, \pm \frac{1}{2}, \pm 1, \pm 2$ and transverse distances $d \equiv |\vec{z}_\perp|/a = 0, 1, 2$. The longitudinal-momentum fractions were set to $f_{1,2} = \frac{1}{2}$ without loss of generality¹¹ and different configurations in the transverse plane were studied, including the one relevant to meson-meson scattering, that is the *average* over the transverse orientations (“*ave*”).

Numerical simulations of LGT provide (within the errors) the true QCD expectation for \mathcal{C}_E ; approximate analytical calculations of \mathcal{C}_E have then to be compared with the lattice data, in order to test the goodness of the approximations involved. \mathcal{C}_E has been evaluated in the *Stochastic Vacuum Model* (SVM),¹² $\mathcal{C}_E^{(\text{SVM})} = \frac{2}{3} e^{-\frac{1}{3} K_S \cot \theta} + \frac{1}{3} e^{\frac{2}{3} K_S \cot \theta} - 1$, in PT,^{8,12,13} $\mathcal{C}_E^{(\text{PT})} = K_p \cot^2 \theta$, in the *Instanton Liquid Model* (ILM),^{11,14} $\mathcal{C}_E^{(\text{ILM})} = \frac{K_1}{\sin \theta}$, and, using the AdS/CFT correspondence, for planar, strongly coupled $\mathcal{N} = 4$ SYM at large $|\vec{z}_\perp|$,¹⁵ $\mathcal{C}_E^{(\text{AdS/CFT})} = e^{\frac{K_1}{\sin \theta} + K_2 \cot \theta + K_3 \cos \theta \cot \theta} - 1$. The coefficients $K_i = K_i(\vec{z}_\perp; \nu_1, \nu_2)$ are functions of \vec{z}_\perp and of the dipole variables $\vec{r}_{i\perp}, f_i$. The comparison of the lattice data with these analytical calculations, performed in Ref.¹⁰ by fitting the lattice data with the corresponding functional form, is not fully satisfactory, even though largely improved best fits have been obtained by combining the ILM and PT expressions into the expression $\mathcal{C}_E^{(\text{ILMp})} = \frac{K_{\text{Ip1}}}{\sin \theta} + K_{\text{Ip2}} \cot^2 \theta$. Regarding the energy dependence of total cross sections, the above analytical models are absolutely unsatisfactory, as they do not lead to *Froissart-like* total cross sections at high energy, as experimental data seem to suggest. Infact, the SVM, PT, ILM and ILMp parameterizations lead to asymptotically constant $\sigma_{\text{tot}}^{(hh)}$, while the AdS/CFT result leads to power-like $\sigma_{\text{tot}}^{(hh)}$.¹⁶

2 How a Froissart-like total cross section can be obtained

One is thus motivated to look for new parameterizations of the CF that: i) fit well the data; ii) satisfy the unitarity condition after analytic continuation; and iii) lead to total cross sections rising as $B \log^2 s$ in the high-energy limit.¹⁷ Regarding unitarity, from (1) and (2) one recognizes that the quantity $A(s, |\vec{z}_\perp|) \equiv \langle \langle \mathcal{C}_M(\chi; \vec{z}_\perp; 1, 2) \rangle \rangle$ is the scattering amplitude in impact-parameter space, which must satisfy the *unitarity constraint* $|A + 1| \leq 1$.¹⁸ Since $\int d\nu_i \rho_i(\nu_i) = 1$, this is the case if the following *sufficient* condition is satisfied: $|\mathcal{C}_M(\chi; \vec{z}_\perp; \nu_1, \nu_2) + 1| \leq 1, \forall \vec{z}_\perp, \nu_1, \nu_2$.

The conditions above constrain rather strongly the possible parameterizations. We shall *assume* that the Euclidean CF can be written as $\mathcal{C}_E = \exp K_E - 1$, where $K_E = K_E(\theta; \vec{z}_\perp; \nu_1, \nu_2)$ is a *real* function (since \mathcal{C}_E is *real*¹⁰). This assumption is rather well justified: in the large- N_c expansion, $\mathcal{C}_E \sim \mathcal{O}(1/N_c^2)$, so that $\mathcal{C}_E + 1 \geq 0$ is certainly satisfied for large N_c ; all the known analytical models satisfy it; the lattice data of Refs.^{10,11} confirm it. The Minkowskian CF is then obtained after analytic continuation: $\mathcal{C}_M = \exp K_M - 1$, with $K_M(\chi; \dots) = K_E(\theta \rightarrow -i\chi; \dots)$. At large χ , \mathcal{C}_M is expected to obey the above-mentioned unitarity condition, which in this case reduces to $\text{Re}K_M \leq 0 \quad \forall \vec{z}_\perp, \nu_1, \nu_2$.

For a *confining* theory like QCD, \mathcal{C}_E is expected to decay exponentially as $\mathcal{C}_E \sim (\sum) e^{-\mu|\vec{z}_\perp|}$ at large $|\vec{z}_\perp|$, with mass scales μ related to the masses of particles (including, possibly, also *glueballs*¹⁹) exchanged between the two WL. Therefore, one also expects a similar large- $|\vec{z}_\perp|$ behavior for K_E , i.e., $K_E \sim (\sum) e^{-\mu|\vec{z}_\perp|}$. (Instead, for a non-confining, let's say *conformal*, field theory, different behaviors like powers of $1/|\vec{z}_\perp|$ are typical.^{15,16})

Let us now assume that the leading term of the Minkowskian CF for $\chi \rightarrow +\infty$ is of the form $\mathcal{C}_M \sim \exp(i\beta f(\chi) e^{-\mu|\vec{z}_\perp|}) - 1$ (i.e., $K_M \sim i\beta f(\chi) e^{-\mu|\vec{z}_\perp|}$) where $\beta = \beta(\nu_1, \nu_2)$ is a function of the dipole variables and $f(\chi)$ is a *real* function such that $f(\chi) \rightarrow +\infty$ for $\chi \rightarrow +\infty$. In this case, the unitarity condition is equivalent (for large χ) to $\text{Im}\beta \geq 0$. By virtue of the *optical theorem*, $\sigma_{\text{tot}}^{(hh)}(s) \sim s^{-1} \text{Im}\mathcal{M}_{(hh)}(s, t=0)$, we find $\sigma_{\text{tot}}^{(hh)} \sim 4\pi\mu^{-2} \langle \langle \frac{1}{2} \log^2 f(\chi) + \log f(\chi)(\log|\beta| + \gamma) + \dots \rangle \rangle$. If one takes $f(\chi) = \chi^p e^{n\chi}$, the resulting asymptotic behavior of $\sigma_{\text{tot}}^{(hh)}$ is [recall $\chi \simeq \log(s/m^2)$]¹⁷

$$\sigma_{\text{tot}}^{(hh)} \sim B \log^2 s, \quad \text{with:} \quad B = \frac{2\pi n^2}{\mu^2}. \quad (3)$$

We want to emphasize that the above result is *universal*, depending only on the mass scale μ , which sets the large- $|\vec{z}_\perp|$ dependence of the leading term of the CF, since the integration over the dipole variables does *not* affect the leading term. The *universal* coefficient B is not affected by the masses of the scattering particles: for mesons of masses $m_{1,2}$, the rapidity becomes $\chi \sim \log(\frac{s}{m_1 m_2})$, which simply corresponds to a change of the energy scale implicitly contained in (3). This result can also be extended to the case in which K_M is, for large χ , the sum of different terms, each behaving like the one discussed above, but with different values of n and μ , i.e., $K_M \sim i \sum_k \beta_k \chi^{p_k} e^{n_k \chi} e^{-\mu_k |\vec{z}_\perp|}$: the resulting B comes out to be determined by the maximum value of the ratio n/μ among these terms, i.e., $B = 2\pi \max_k (\frac{n_k}{\mu_k})^2$. From a physical point of view, one expects that particles with *mass* M and *spin* J , exchanged between the two WL, contribute with $\mu = M$ and $n = J - 1$: in fact, in this case the factor $e^{n\chi}$ reduces to the well-known factor s^{J-1} , expected for the contribution of an exchanged particle of spin J .²⁰

3 New analysis of the lattice data

In Ref.¹⁷ we have found three parameterizations $\mathcal{C}_E^{(i)} = \exp K_E^{(i)} - 1$, $i = 1, 2, 3$ (among the many that we have analyzed), that satisfy the criteria i)–iii) listed above. We have focused our analysis on the *averaged* CF \mathcal{C}^{ave} , that is “closer” to the hadronic scattering matrix $\mathcal{M}_{(hh)}$.

The first two parameterizations, $K_E^{(1)} = \frac{K_1}{\sin\theta} + K_2 \cot^2\theta + K_3 \cos\theta \cot\theta$ and $K_E^{(2)} = \frac{K_1}{\sin\theta} + K_2(\frac{\pi}{2} - \theta) \cot\theta + K_3 \cos\theta \cot\theta$, are essentially two proper modifications of the AdS/CFT result. The third parameterization is, instead, $K_E^{(3)} = \frac{K_1}{\sin\theta} + K_2(\frac{\pi}{2} - \theta)^3 \cos\theta$: while the first term is “familiar”, the second one is not present in the known analytical models, but it is a fact that the resulting best fit is extremely good. In the three cases, the unitarity condition $\text{Re}K_M^{(i)} \leq 0$ is satisfied if $K_2 \geq 0$: this is actually the case for our best fits (within the errors). The leading term after analytic continuation (the third term in the first two parameterizations $K^{(1)}$ and $K^{(2)}$ and the second term in the parameterization $K^{(3)}$) is of the form $\chi^p e^\chi$ for large χ , which, according to the previous discussion, should correspond to an exchanged particle of spin $J = 2$ (being $n = 1$), and, according to (3), leads to $\sigma_{\text{tot}}^{(hh)} \sim B \log^2 s$. The value of $B = 2\pi/\mu^2$, obtained through a fit of the coefficient of the leading term with an exponential function $\sim e^{-\mu|\vec{z}_\perp|}$ over the available distances, is found to be compatible with the experimental result $B_{\text{exp}} \simeq 0.3 \text{ mb}$

Table 1: Mass-scale μ , “decay length” $\lambda = 1/\mu$ and the coefficient $B = 2\pi/\mu^2$ obtained with our parameterizations.

	μ (GeV)	$\lambda = \frac{1}{\mu}$ (fm)	$B = \frac{2\pi}{\mu^2}$ (mb)
Corr 1	4.64(2.38)	$0.042^{+0.045}_{-0.014}$	$0.113^{+0.364}_{-0.037}$
Corr 2	3.79(1.46)	$0.052^{+0.032}_{-0.014}$	$0.170^{+0.277}_{-0.081}$
Corr 3	3.18(98)	$0.062^{+0.028}_{-0.015}$	$0.245^{+0.263}_{-0.100}$

(within the large errors) in all the three cases (see Table 1). However, this must be taken only as an estimate, as lattice data are available only for small $|\vec{z}_\perp|$. Since $\mu_{\text{exp}} = \sqrt{2\pi/B_{\text{exp}}} \simeq 2.85$ GeV is close to the value of the mass for the *glueball* 2^{++} ,¹⁹ one could be tempted to conclude, on the basis of the results that we have found, that B is determined by the mass of this *glueball*. But this is maybe still a premature conclusion: work is in progress along this direction.²⁰

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