One-loop multi-leg calculations in gauge theories: the Golem library

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I present a program designed for the numerical evaluation of one-loop multi-leg scattering amplitudes. This program, called `Golem95`, is written in `FORTRAN-95`, and it contains all the form factors involved in one-loop calculations of amplitudes with up to six external legs, for arbitrary internal masses (complex masses are supported). This library is based on the traditional algebraic reduction (Golem reduction) of the form factors into a certain redundant set of basis integrals; this reduction formalism is able to avoid the problem of the Gram determinant spurious singularity. `Golem95` can be used to calculate one-loop amplitudes in the framework of an algebraic reduction approach, as well as in the framework of inspired-unitarity reconstruction of the amplitudes (`GoSam`) [1].

1 Introduction

Higher order corrections in gauge theories play a crucial role in studying physics within the standard model and beyond at TeV colliders, like Tevatron, LHC and ILC. Therefore, it is of extreme importance to provide a tool for next-to-leading order (NLO) computation which is fast, stable, efficient in a highly automated way.

In the last few years, an important progress in the automation of multi-leg one-loop scattering amplitude calculations have been made. Various programs and libraries have been developed for this task, many of them are publicly available.

The library of one-loop integrals is one of the main ingredient of any NLO automatic program. In this article, I present the `Golem95` library, which is a program written in `FORTRAN-95`, designed for one-loop calculations of amplitudes with up to six external legs, for arbitrary internal masses (complex masses are supported). This library is based on the traditional algebraic reduction (Golem reduction) of the form factors into a certain redundant set of basis integrals; this reduction formalism is able to avoid the problem of the Gram determinant spurious singularity ($\det(G)$). In addition, `Golem95` can be used to calculate one-loop amplitudes in the framework of inspired-unitarity reduction method, and it can be used as library of master integrals.

2 Golem algebraic reduction

The Golem reduction formalism [2, 3] is designed to express any one-loop N-point Feynman diagram, up to six external legs, as a linear combination in term of a set of redundant basic integrals and given coefficients without facing any inverse of Gram determinants. A general N-point tensor integral of rank r in dimension $n = 4 - 2\,\epsilon$ is defined as

$$I_N^{n;\mu_1,\ldots,\mu_r}(a_1,...,a_r;S) \;=\; \int \frac{d^n q}{i\,\pi^{n/2}} \frac{q_{a_1}^{\mu_1}\cdots q_{a_r}^{\mu_r}}{\Pi_{i\in S}^N\,(q_i^2 - m_i^2 + i\,\delta)} \tag{1}$$

where $q_i = q + r_i$, q is the loop momentum, r_i is a combination of external momenta and m_i is the mass of the internal line "i", S is an ordered set containing the labels of the propagators.

2.1 Form factors

The tensor integral presented above can be expressed as a linear combination of such Lorentz tensors and scalar quantities "$A^{N,r}$, $B^{N,r}$, $C^{N,r}$" called form factors, i.e. it can be written in the following form

$$\begin{aligned} I_N^{n;\mu_1,\ldots,\mu_r}(a_1,...,a_r;S) \;=\; & \sum_{j_1\cdots j_r\in S} [\Delta_{j_1\bullet}^{\bullet}\cdots\Delta_{j_r\bullet}^{\bullet}]_{\{a_1\cdots,a_r\}}^{\{\mu_1\cdots\mu_r\}}\, A_{j_1\cdots j_r}^{N,r}(S) \\ & + \sum_{j_1\cdots j_{r-2}\in S} [g^{\bullet\bullet}\,\Delta_{j_1^\bullet}^{\bullet}\cdots\Delta_{j_{r-2}\bullet}^{\bullet}]_{\{a_1\cdots,a_r\}}^{\{\mu_1\cdots\mu_r\}}\, B_{j_1\cdots j_{r-2}}^{N,r}(S) \\ & + \sum_{j_1\cdots j_{r-4}\in S} [g^{\bullet\bullet}\,g^{\bullet\bullet}\,\Delta_{j_1\bullet}^{\bullet}\cdots\Delta_{j_{r-4}\bullet}^{\bullet}]_{\{a_1\cdots a_r\}}^{\{\mu_1\cdots\mu_r\}}\, C_{j_1\cdots,j_{r-4}}^{N,r}(S) \end{aligned} \tag{2}$$

the shift invariant vector Δ_{ij}^{μ} is defined as difference of the two propagator momenta q_i and q_j. $A^{N,r}$ is the coefficient of the Lorentz structure containing only these vectors. $B^{N,r}$ and $C^{N,r}$ are the coefficients of the Lorentz tensors containing one and two metric tensor, respectively. The square brackets $[\cdots]_{\{a_1\cdots,a_r\}}^{\{\mu_1\cdots\mu_r\}}$ are interpreted as the distribution of the r Lorentz indices μ_i, and the momentum labels a_i in all distinguishable ways to the vectors $\Delta_{ja_i}^{\mu_i}$ and the metric tensors. As an example, the scalar, the tensor of rank one and the tensorial of rank two N-point integrals can be written, respectively, in the following form

$$\begin{aligned} I_N^n(S) \;&=\; A^{N,0}(S) \qquad\qquad I_N^{n,\mu_1}(a_1;S) = \sum_{l\in S} \Delta_{l\,a_1}^{\mu_1}\, A_l^{N,1}(S) \\ I_N^{n,\mu_1\,\mu_2}(a_1,a_2;S) \;&=\; \sum_{l_1,l_2\in S} \Delta_{l_1\,a_1}^{\mu_1}\,\Delta_{l_2\,a_2}^{\mu_2}\, A_{l_1 l_2}^{N,2}(S) + g^{\mu_1\,\mu_2}\, B^{N,2}(S) \end{aligned} \tag{3}$$

After Feynman parameterization, one can prove that any of these form factors can be expressed in term of integrals without Feynman parameters in the numerator called scalar integrals, and integrals with Feynman parameters in the numerator called tensorial integrals. These integrals can be reduced to some set of basic integrals using the traditional algebraic approach.

2.2 Reduction by subtraction

The scalar one-loop integral $I_N^n(S)$ is obtained from Eq. (1), by replacing the numerator by 1; this integral can be split into an IR and a finite part by making the ansatz

$$\begin{aligned} I_N^n(S) \;&=\; \sum_{i\in S} b_i(S) \int \frac{d^n q}{i\,\pi^{n/2}} \frac{(q_i^2 - m_i^2)}{\Pi_{j\in S}\,(q_j^2 - m_j^2 + i\,\delta)} + \int \frac{d^n q}{i\,\pi^{n/2}} \frac{1 - \sum_{i\in S} b_i(S)(q_i^2 - m_i^2)}{\Pi_{j\in S}\,(q_j^2 - m_j^2 + i\,\delta)} \\ &=\; I_{div}(S) + I_{fin}(S) \end{aligned} \tag{4}$$

the b_i are fixed in such way that this integral is reduced to an IR divergent integral in n-dimension with one less propagator I_{div}, and a finite integral in $n+2$-dimension with the same number of propagators I_{fin}. By repeating this procedure several times, ultimately any Feynman diagram is expressed in term of integrals up to 4-point [a].

2.3 Basis integrals

Using Golem reduction formalism, the one-loop integrals and consequently the form factors are reduced to a set of redundant basic integrals. This set of integrals does not form a basis in the mathematical sense, but it contains the end point of our reduction. This reduction formalism plays a crucial rule on the numerical stability of `Golem95` in some region of phase space where the Gram determinant becomes arbitrary small; thanks to the choice of basic integrals which avoid these spurious Gram singularities.
It turns out that, the set of basic integrals that allows us to express any one-loop amplitude up to 6-external legs without producing any spurious Gram determinant singularities consists of the following scalar and tensorial integrals:

$$\{I_2^n(l_i;S), I_3^n(l_i;S), I_3^{n+2}(l_i;S), I_4^{n+2}(l_i;S), I_4^{n+4}(l_i;S)\}$$

l_i stands for Feynman parameters. This basis contains: the 4-point functions in $n+2$, which are IR and UV finite, the 4-point functions in $n+4$ dimensions which are UV divergent, the 3-point functions in $n+2$ dimensions, the 3-point functions in n dimension where all possible IR divergences are isolated, and various two point functions.

3 Avoiding spurious Gram singularities

A further reductions of the tensorial elements of Golem basis to a scalar integral leads to expressions containing inverses $\det(G)$ [b]. The philosophy of `Golem95` to avoid such spurious singularities is the following:
we provide for each basic integral, an analytical formula and a one-dimensional integral representation. The former representation is obtained by performing all the integrations analytically; and the later one is obtained by performing the first integrations analytically and keeping the last one, which will be performed numerically after modifying the integrand such that the inverse of Gram determinant is avoided.
Lets focus on the 4-point basic integrals. We notice that before modifying the integrand of the one-dimensional integral representation, we encounter terms have the following form:

$$\int dy \, \frac{\ln(1-F(y,x_\pm) + \sum_{i=1}^{m-1} \frac{F(y,x_\pm)^{m-i}}{m-i}}{F(y,x_\pm)^m}, \qquad m = 1,2,3,4 \tag{5}$$

$x_\pm$ are the two solutions of a given quadratic equation, where the associated discriminant is proportional to the Gram determinant associated to the triangle (obtained by pinching a given propagator of the box) and $F(y,0)=0$. Since the box Gram determinant $\det(G) \propto x_+ \, x_-$, the function F vanishes if $\det(G)$ vanishes. This means that the term given in Eq.(5) is not any more numerically stable in this configuration. To deal with this problem, we write Eq.(5) in the following form

$$-\int dy \sum_{n=0}^{\infty} \frac{F(y,x_\pm)^n}{n+m} \tag{6}$$

This Taylor expansions provide a numerical stable results for $x_\pm \to 0$, hence for $\det(G) \to 0$.
To summarize, the strategy of Golem95 to avoid Gram determinant spurious singularities is:
- a) if $\det(G)$ is large, the basic integrals are evaluated analytically, which provide a fast and

[a] For $N=5$ and $N \geq 6$, the N-point finite parts I_{fin} are absent at one-loop order.
[b] For example, the redundant integral I_4^{n+2} can be expressed in term of the master integrals as the following:
$I_4^{n+2}(S) \;=\; \det(S)/\det(G)\,[I_4^n(S) - \sum_{i=1}^4 \, b_i \, I_3^n(S \setminus \{i\})] \;\propto\; 1/\det(G)$

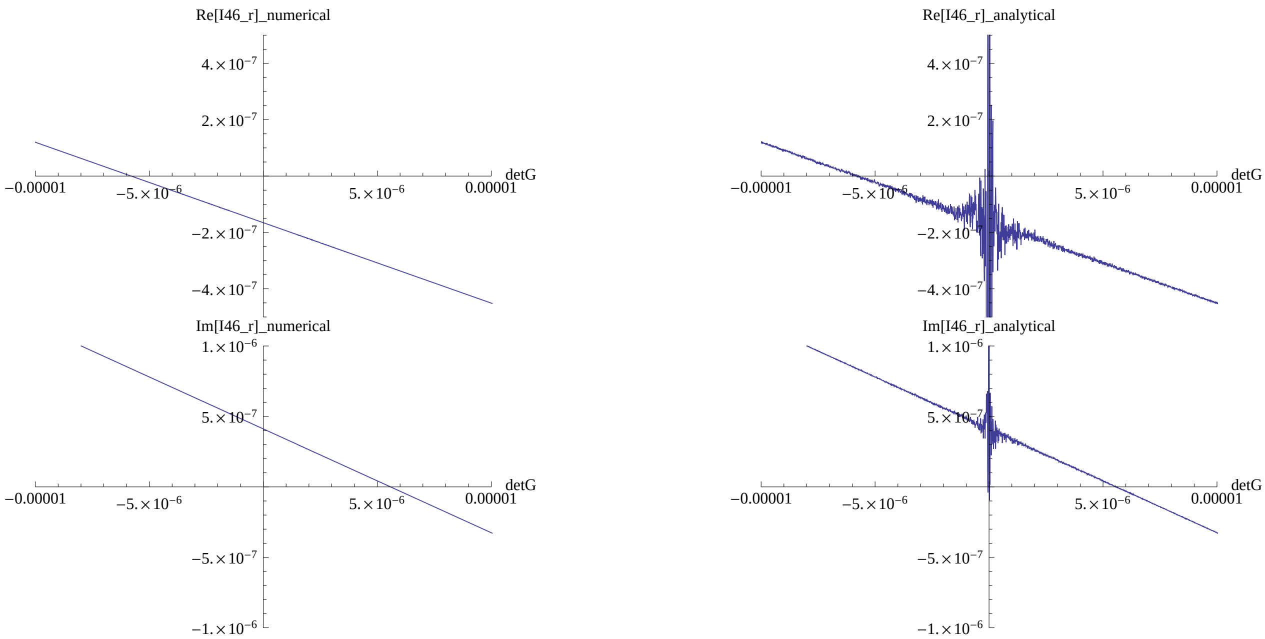

Figure 1: I_4^{n+2} versus $\det(G)$, Comparison of the numerical and the analytical modes

efficient numerical evaluation in large phase space region
- b) if $\det(G)$ is small, we switch to the numerical mode by integrating the one-dimensional representation of each basic integral numerically
In Figure 1, a comparison between the numerical and the analytical modes of evaluating the redundant basic integral $I_4^6(S)$. The two plots in the right hand side, represent the real (up) and the imaginary (down) parts of a normalized I_4^6 in the analytical mode, versus $\det(G)$. These plots show fluctuations of the function in the region $\det(G) \to 0$, which means that the result is not any more stable. However, in the numerical mode (plots in the left hand side represent the same function evaluation numerically) the function is smooth near and for $\det(G) = 0$, i.e. it is numerically stable.

4 Conclusion

I have presented the library `Golem95` for the numerical evaluation of tensor integrals up to 6-point functions, valid for arbitrary internal masses. The library is based on Golem reduction method to reduce the form factor to a certain set of basic integrals; this reduction formalism is able to avoid the problems occurring by spurious Gram singularities. The basic integrals are implemented in both analytical and one-dimensional integral forms; if the Gram determinant is large the program uses the former form, otherwise it switches to the numerical integration of the later form. `Golem95` can be used as a library of master integrals as the scalar integrals are related directly to form factors without Feynman parameters labels. It can be used also to calculate one-loop amplitudes in the framework of inspired-unitarity reduction (`GoSam`).

References

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